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**Solution to a Forced and Damped Gardner equation by means of Cubic B-splines**

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**Abstract**

In this paper, we make use of cubic splines to solve the forced and damped Gardner equation. We illustrate the obtained results by a concrete example. We also derive approximate analytical solution for the damped or forced case.

# 1 Introduction

Finding a numerical solution of a PDE is a computationally intensive task in many scientific and engineering applications. Even “routine” modeling problems such as weather forecasting can be extremely challenging in practice and the development of fast and accurate PDE solvers is an important field of research.

Finite-element numerical methods (FEM) for PDE's rely on mesh-based domain discretization and employ polynomial basis functions. FEM has been studied extensively for decades and became the de-facto instrument for solving PDE's on arbitrarily-shaped domains. The spline functions are used extensively to solve initial and boundary value problems [1], [2]. These functions preserve smoothness at the nodes and have the ability to provide the numerical solution in the entire domain with great accuracy.

**Key words and phrases:** Cubic splines, Gardner equation, forced Gardner, damped Gardner.
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In this paper, we make use of cubic B-splines in order to solve the damped and forced Gardner equation

$$\begin{array}{l} u_t + (\alpha u + \beta u^2)u_x + \beta u_{xxx} + Ru = F_0 \cos(\omega t),\ u(x,0) = f(x). \\ a < x < b,\ T_0 < t < T_1,\ T_0 \leq 0 < T_1. \end{array} \tag{1.1}$$

The Gardner equation has two nonlinear terms in quadratic and cubic forms and the dissipative term is of third order. The Gardner equation is an integrable system and Miura transformation connects it to the KdV equation [1]. The Gardner equation is a useful model to understand the propagation of negative ion acoustic plasma waves [2]. The equation can be derived from the system of plasma motion equations in one dimension with arbitrarily charged cold ions and inertia neglected isothermal electrons. The Gardner equation can also be a good description of internal waves with large amplitudes [3].

## 2 Odd order B splines

The general odd B-splines of order $2r-1$ are defined as

$$\begin{aligned} \varphi_i(x) &= h^{-(2r-1)} \\ & \left( \begin{array}{c} \sum_{j=0}^{r-1} \left( \sum_{k=0}^{j} (-1)^k \binom{2r}{k} \left(x - \xi_{i-(r-k)}\right)^{2r-1} \right) \chi_{i-(r-j)}(x) + \\ \sum_{j=0}^{r-1} \left( \sum_{k=0}^{j} (-1)^k \binom{2r}{k} \left(\xi_{i+r-k} - x\right)^{2r-1} \right) \chi_{i+(r-j)-1}(x) \end{array} \right), \end{aligned}$$

$$\begin{aligned} \text{where } h &= \frac{b-a}{n},\ \xi_i = a + ih \text{ and } \chi_s = \chi_{[a+sh, a+(s+1)h)}, \text{ that is} \\ \chi_s(x) &= 1 \text{ if } a + sh \leq x < a + (s+1)h \text{ and } 0 \text{ otherwise :} \end{aligned}$$

When $r = 2$ we obtain the so called cubic B-splines as follows :

$$\varphi_i(x) = h^{-3} \left( \begin{array}{c} \left(x - \xi_{i-2}\right)^3 \chi_{i-2}(x) + \left(\left(x - \xi_{i-2}\right)^3 - 4\left(x - \xi_{i-1}\right)^3\right) \chi_{i-1}(x) + \\ \left(\left(\xi_{i+2} - x\right)^3 - 4\left(\xi_{i+1} - x\right)^3\right) \chi_i(x) + \left(\xi_{i+2} - x\right)^3 \chi_{i+1}(x) \end{array} \right)$$

| * | $\varphi_i$ | $\varphi_i{}'$ | $\varphi_i{}''$ | $\varphi_i{}^{(3)}$ |
|---|---|---|---|---|
| $\xi_{i-2}$ | 0 | 0 | 0 | $\frac{6}{h^3}$ |
| $\xi_{i-1}$ | 1 | $\frac{3}{h}$ | $\frac{6}{h^2}$ | $-\frac{18}{h^3}$ |
| $\xi_i$ | 4 | 0 | $-\frac{12}{h^2}$ | $\frac{18}{h^3}$ |
| $\xi_{i+1}$ | 1 | $-\frac{3}{h}$ | $\frac{6}{h^2}$ | $-\frac{6}{h^3}$ |

**Table 1**

Assuming that $u(x,t) = \sum_{k=i-1}^{n+1} \delta_k(t)\varphi_k(x)$ and with the help of Table 1, we get

$$\begin{aligned}
&u_t(\xi_i,t) = h^{-6}\left(\delta'_{i-1}(t) + 4\delta'_i(t) + \delta'_{i+1}(t)\right).\\
&u(\xi_i,t) = h^{-6}\left(\delta_{i-1}(t) + 4\delta_i(t) + \delta_{i+1}(t)\right).\\
&u_x(\xi_i,t) = 3h^{-5}\left(\delta_{i+1}(t) - \delta_{i-1}(t)\right).\\
&u_{xx}(\xi_i,t) = 6h^{-4}\left(\delta_{i-1}(t) - 2\delta_i(t) + \delta_{i+1}(t)\right).\\
&u_{xxx}(\xi_i,t) = 6h^{-3}\left(-\delta_{i-1}(t) + 3\delta_i(t) - 3\delta_{i+1}(t) + \delta_{i+2}(t)\right).
\end{aligned} \tag{2.2}$$

These formulas may be employed to solve pde's like KdV, KdV-Burgers, MKdV, Gardner and many third order pde's arising in physics.

# 3 The forced and damped KdV equation and its solution

Suppose that $v = v(x,t)$ is a solution to the Gardner equation $u_t + (\alpha u + \beta u^2)u_x + \beta u_{xxx} = 0$. Assume that $u = \exp(-Rt)v(x \cdot g(t), h(t))$. Then

$$\begin{array}{l} u_t + \alpha u u_x + \beta u_{xxx} + Ru - F_0\cos(\omega t) = \\ e^{-Rt} x g'(t) v_x + e^{-3Rt} v^2 \beta \left(g(t) - e^{2Rt} h'(t)\right) v_x + \\ e^{-Rt}\gamma\left(g(t)^3 - h'(t)\right) v_{3x} + e^{-Rt} g(t) v_x \phi(t)(\alpha + \beta\phi(t)) + \\ e^{-2Rt} v v_x \left(-e^{Rt}\alpha h'(t) + g(t)(\alpha + 2\beta\phi(t))\right) + [\phi'(t) + R\phi(t) - F_0\cos(\omega t)] \end{array} \tag{3.3}$$

We will make the choices $h'(t) = e^{-3Rt}$, $g(t) = e^{2Rt}h'(t)$, $h(0) = 0$, $\phi'(t) + R\phi(t) - F_0\cos(\omega t) = 0$ and $\phi(0) = 0$. The needed expressions are:

$$\begin{array}{l} g(t) = \exp(-Rt),\ h(t) = \frac{1}{3R}\left(1 - \exp\left(-3Rt\right)\right). \\ \text{and } \phi(t) = \frac{F_0}{R^2+\omega^2}\left(R\cos(\omega t) + \omega\sin(\omega t) - e^{-Rt}R\right). \end{array} \tag{3.4}$$

For these choices we will have the following residual error:

$$\text{Residual} = ve^{-3Rt}(\alpha + 2\beta\phi(t))v_x + e^{-2Rt}\left(\phi(t)(\alpha + \beta\phi(t))v_x - Rxv_x\right) - \alpha v e^{-4Rt} v_x.$$

In order to solve the i.v.p. (1.1), we must solve the following system of nonlinear odes:

$$\begin{array}{l} \delta'_{i-1}(t) + 4\delta'_i(t) + \delta'_{i+1}(t) - \frac{6\gamma(\delta_{i-1}(t) - 3\delta_i(t) + 3\delta_{i+1}(t) - \delta_{i+2}(t))}{h^3} - \\ \frac{3}{h}\left[\begin{array}{c} (\delta_{i-1}(t) - \delta_{i+1}(t))\left(\delta_{i-1}(t) + 4\delta_i(t) + \delta_{i+1}(t)\right) \\ \left(\alpha + \beta\left(\delta_{i-1}(t) + 4\delta_i(t) + \delta_{i+1}(t)\right)\right) \end{array}\right] = F_0\cos(\omega t). \\ i = -1, 0, ..., n, n+1,\ \delta_j(t) = 0 \text{ for } j < -1 \text{ or } j > n+1. \end{array} \tag{3.5}$$

The initial conditions for the odes in (3.3) are obtained from the linear system $u(\xi_i, 0) = f(\xi_i)$, $i = -1, 0, 1, 2, ..., n, n+1$. We must choose the value of $h = (b-a)/n$ in order to get the least residual error as possible. Usually, this number lies on the interval $[1, p]$ for some $p > 1$. Choosing too small value for $h$ may give a heavily oscillatory system of odes and this will give bad results.

# 4 Analysis and Discussion

We have obtained approximate analytical solution to the damped and forced Gardner equation starting form an exact solution to undamped and unforced

Gardner equation. Some exact solutions to the Gardner equation $u_t + (\alpha u + \beta u^2)u_x + \beta u_{xxx} = 0$ are

$$u_{\text{cnoidal}}(x,t) = -\frac{\beta}{2\alpha} \pm k\sqrt{\frac{6\gamma m}{\alpha}}\text{cn}\left(kx + \frac{\beta^2 k - 4\alpha\gamma k^3\,(2m-1)}{4\alpha}t + \xi_0, m\right). \tag{4.6}$$

$$u_{\text{soliton}}(x,t) = --\frac{\beta}{2\alpha} \pm k\sqrt{\frac{6\gamma}{\alpha}}\text{sech}\left(kx + \frac{\beta^2 k - 4\alpha\gamma k^3}{4\alpha}t + \xi_0\right) \tag{4.7}$$

Let us consider an example.

**Example**. Let

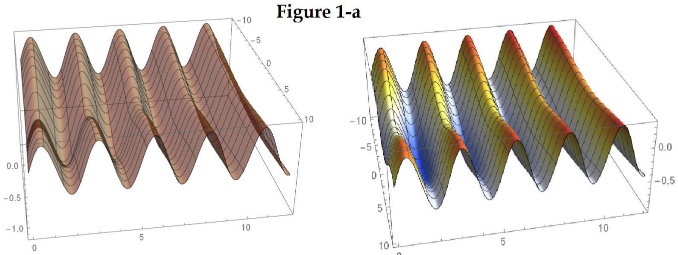
$$u_t + (u + 3u^2)u_x + u_{xxx} + 0.1u = \cos 2.5t,$$
$$u(x,0) = -0.740644\text{sech}(0.523715x) - 0.166667.,\ -10 < x < 10,\ 0 < t < 12.$$

The approximate analytical solution is given by

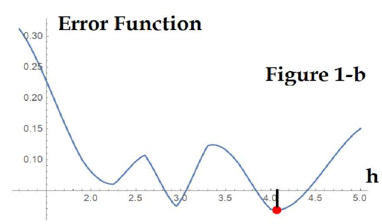
$$u_{\text{approx}}(x,t) = -0.740644e^{-0.1t}\text{sech}\left(-0.523715e^{-0.1t}x - 0.333333e^{-0.3t} + 0.333333\right) - 0.182641e^{-0.1t} + 0.399361\sin(2.5t) + 0.0159744\cos(2.5t).$$

The approximate analytical solution and the cubic splines solution are depicted in Figure 1-a. In Figure 1-b, we show the plot of the global residual error as a function of $h$. From that plot, we see that the optimal value for $h$ is $h = 4.1$ and then $n = 4$. The global error for the analytical approximant equals 0.22 and the error for the cubic splines equals 0.03.

**Figure 1-a**



**Analytical Approximant** **Cubic Splines**




**Optimal Value for h**

# References


[1] Alvaro H. Salas, Computing solutions to a forced KdV equation Nonlinear Analysis, Real World Applications **12**, no. 2, (2011), 1314–1320.

[2] Alvaro H. Salas Analytical Approximant for a Damped KdV Equation, Int. J. Math.Comput. Sci., **16,** no. 4, (2021), 1631–1636.