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On Some Absolute Matrix Trace Inequalities

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Abstract

In this paper, we present some trace inequalities for the absolute of product matrices by applying inequalities of singular value and eigenvalue of product matrices.

1 Introduction

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices over the complex number field $\mathbb{C}$. The singular values of $A \in M_n(\mathbb{C})$ denoted by $\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)$

are the eigenvalues of $|A| = \Big(A^*A\Big)^{1/2}$. All singular values of the matrix A are arranged in decreasing order as $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0$. In particular, when the eigenvalues of A are real numbers, let its eigenvalues satisfy $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. Note that $\sigma_i^2(A) = \lambda_i(A^*A) = \lambda_i(AA^*)$ so for a positive semidefinite matrix A we have $\sigma_i(A) = \lambda_i(A)$.

Given two real vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in decreasing order, we say that x is weakly log majorized by y, denoted by $x \prec_{wlog} y$, if $\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i$, $k = 1, 2, \ldots, n$ and we say that x is weakly majorized by y, denoted by $x \prec_w y$, if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $k = 1, 2, \ldots, n$.

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We say x is majorized by y, denoted by $x \prec y$, if

$$x \prec_w y \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

The weak majorization of the product of singular values for $A, B \in M_n(\mathbb{C})$

$$\sum_{j=1}^{n} \sigma_j(AB) \leq \sum_{j=1}^{n} \sigma_j(A)\sigma_j(B) = \sum_{j=1}^{n} \lambda_j(|A|)\lambda_j(|B|).$$

The trace function of $A \in M_n(\mathbb{C})$, denoted by $\mathrm{tr}(A,)$ is defined to be the sum of the all main diagonal entries of A. It is well known that the trace of matrix is equal to the sum of all its eigenvalues; that is, $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i(A)$. Trace inequalities are used in many applications in mathematics such as in adaptive stochastic control and for the investigation of quantum mechanical Hamiltonians and are explicitly presented by Patel and Toda [6], [7] and [5] and Lieb and Thirring [4]. For the theory of trace function and their applications, we refer the reader to [3].

The main purpose of this paper is to establish trace inequalities for absolute matrices.

2 Main results

Throughout this section, we work with square matrices on complex numbers.

Theorem 2.1. *Let $A_i, B_i \in M_n(\mathbb{C})$ and let p, q be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\mathrm{tr}\left(\sum_{i=1}^{m} |A_i B_i|\right) \leq \sum_{i=1}^{m} \mathrm{tr}^{1/p}\left(|A_i|^p\right) \sum_{i=1}^{m} \mathrm{tr}^{1/q}\left(|B_i|^q\right).$$

Proof. For any $i = 1, \ldots, m$, it is well known that $\mathrm{tr}(|A_i B_i|) = \sum_{j=1}^{n} \lambda_j(|A_i B_i|)$. Since the singular values for any matrix are the eigenvalues of its absolute value,

$$\mathrm{tr}(|A_i B_i|) = \sum_{j=1}^{n} \sigma_j(A_i B_i).$$

By the weak majorization of the product of singular values, we have

$$\sum_{j=1}^{n} \sigma_j(A_i B_i) \leq \sum_{j=1}^{n} \sigma_j(A_i)\sigma(B_i) = \sum_{j=1}^{n} \lambda_j(|A_i|)\lambda(|B_i|).$$

Using Hölder's inequality for positive real numbers

$$\sum_{j=1}^{n}\lambda_j(|A_i|)\lambda(|B_i|) \le (\sum_{j=1}^{n}\lambda_j^p(|A_i|))^{1/p}(\sum_{j=1}^{n}\lambda_j^q(|B_i|))^{1/q}$$
$$= \operatorname{tr}^{1/p}(|A_i|^p)\operatorname{tr}^{1/q}(|B_i|^q).$$

Then $\operatorname{tr}(\sum_{i=1}^{m}|A_iB_i|) \le \sum_{i=1}^{m}\operatorname{tr}^{1/p}\left(|A_i|^p\right)\sum_{i=1}^{m}\operatorname{tr}^{1/q}\left(|B_i|^q\right)$. □

Theorem 2.2. *Let* $A, B \in M_n(\mathbb{C})$. *Then* $\operatorname{tr}(|AB|^m) \le (\operatorname{tr}(|A|^{2m}))^{1/2}(\operatorname{tr}(|B|^{2m}))^{1/2}$.

Proof. By the weak majorization of the product of singular values

$$\sum_{i=1}^{n}\sigma_i(AB) \le \sum_{i=1}^{n}\sigma_i(A)\sigma_i(B).$$

Thus

$$\operatorname{tr}(|AB|^m) = \sum_{i=1}^{n}\sigma_i^m(AB) \le \sum_{i=1}^{n}\sigma_i^m(A)\sigma_i^m(B) = \sum_{i=1}^{n}\lambda_i^m(|A|)\lambda_i^m(|B|).$$

Using Höder's inequality for positive numbers

$$\sum_{i=1}^{n}\lambda_i^m(|A|)\lambda_i^m(|B|) \le (\sum_{i=1}^{n}\lambda_i^{2m}(|A|))^{1/2}(\sum_{i=1}^{n}\lambda_i^{2m}(|B|))^{1/2}$$
$$= (\operatorname{tr}(|A|^{2m}))^{1/2}(\operatorname{tr}(|B|^{2m}))^{1/2}.$$

So the theorem follows. □

Lemma 2.3. *Let* $A \in M_n(\mathbb{C})$. *Then* $(\operatorname{tr}(|A|^2))^{1/2} \le \operatorname{tr}(|A|)$.

Proof. Since $|A|$ is semi-positive definite, all eigenvalues of $|A|$ are greater than or equal to zero. So

$$(\operatorname{tr}|A^2|)^{1/2} = (\sum_{i=1}^{n}\lambda_i^2(|A|))^{1/2} \le \sum_{i=1}^{n}(\lambda_i^2(|A|))^{1/2} = \operatorname{tr}(|A|).$$

□

Lemma 2.4. *Let* $A \in M_n(\mathbb{C})$. *Then* $|\operatorname{tr}(A)| \le \operatorname{tr}(|A|)$.

Proof. Using Weyl's inequality between eigenvalues and singular values, we have

$$|\operatorname{tr}(A)| = |\sum_{i=1}^{n} \lambda_i(A)| \le \sum_{i=1}^{n} |\lambda_i(A)| \le \sum_{i=1}^{n} \sigma_i(A) = \sum_{i=1}^{n} \lambda_i(|A|) = \operatorname{tr}(|A|).$$

□

Theorem 2.5. *Let $A_i, B_i \in M_n(\mathbb{C})$ and let p, q be positive numbers such that $\frac{1}{p} + \frac{1}{q}$. Then*

$$\Big|\operatorname{tr}(\sum_{i=1}^{m} A_i B_i)\Big| \le \frac{1}{p} \operatorname{tr}\Big(\sum_{i=1}^{m} |A_i|^p\Big) + \frac{1}{q} \operatorname{tr}\Big(\sum_{i=1}^{m} |B_i^*|^q\Big).$$

Proof. Since the trace of matrix is equal to the sum of its eigenvalues,

$$\Big|\operatorname{tr}(\sum_{i=1}^{m} A_i B_i)\Big| \le \operatorname{tr}\Big(\Big|\sum_{i=1}^{m} A_i B_i\Big|\Big) = \sum_{j=1}^{n} \lambda_j(\Big|\sum_{i=1}^{m} A_i B_i\Big|) = \sum_{j=1}^{n} \sigma_j(\sum_{i=1}^{m} A_i B_i).$$

By the Fan singular value majorization theorem [1],

$$\sum_{j=1}^{n} \sigma_j(\sum_{i=1}^{m} A_i B_i) \le \sum_{j=1}^{n} \sum_{i=1}^{m} \sigma_j(A_i B_i),$$

and using the matrix of Young's inequality [8], we have

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \sigma_j(A_i B_i) \le \sum_{j=1}^{n} \sum_{i=1}^{m} \sigma_j\Big(\frac{|A_i|^p}{p} + \frac{|B_i^*|^q}{q}\Big).$$

Again using Fan singular value majorization theorem

$$\begin{aligned}
\sum_{j=1}^{n} \sum_{i=1}^{m} \sigma_j\Big(\frac{|A_i|^p}{p} + \frac{|B_i^*|^q}{q}\Big) &\le \sum_{j=1}^{n} \sum_{i=1}^{m} \Big(\sigma_j\Big(\frac{|A_i|^p}{p}\Big) + \sigma_j\Big(\frac{|B_i^*|^q}{q}\Big)\Big) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m} \Big(\lambda_j\Big(\frac{|A_i|^p}{p}\Big) + \lambda_j\Big(\frac{|B_i^*|^q}{q}\Big)\Big) \\
&= \frac{1}{p} \operatorname{tr}\Big(\sum_{i=1}^{m} |A_i|^p\Big) + \frac{1}{q} \operatorname{tr}\Big(\sum_{i=1}^{m} |B_i^*|^q\Big)
\end{aligned}$$

and the proof is complete. □

Proposition 2.6. *Let* $A, B \in M_n(\mathbb{C})$. *Then* $\operatorname{tr}(|AB|) \leq \operatorname{tr}(|A|)\operatorname{tr}(|B|)$.

Proof. It is known that the trace of an absolute matrix is equal to the sum of its singular values and the singular values for any matrix are the eigenvalues of its absolute value,

$$\operatorname{tr}\Big(|AB|\Big) = \sum_{i=1}^{n} \sigma(AB) \leq \sum_{i=1}^{n} \sigma(A)\sigma(B) = \sum_{i=1}^{n} \lambda(|A|)\lambda(|B|).$$

Now Hölder's inequality for positive numbers implies

$$\sum_{i=1}^{n} \lambda(|A|)\lambda(|B|) \leq \Big(\sum_{i=1}^{n} \lambda(|A|^2)\Big)^{1/2}\Big(\sum_{i=1}^{n} \lambda(|B|^2)\Big)^{1/2} = \Big(\operatorname{tr}(|A|^2)\Big)^{1/2}\Big(\operatorname{tr}(|B|^2)\Big)^{1/2}.$$

Using lemma 2.3 $\Big(\operatorname{tr}(|A|^2)\Big)^{1/2}\Big(\operatorname{tr}(|B|^2)\Big)^{1/2} \leq \operatorname{tr}(|A|)\operatorname{tr}(|B|)$. This completes the proof. □

Corollary 2.7. *Let* $A_i \in M_n(\mathbb{C})$. *Then* $\Big|\operatorname{tr}\Big(\prod_{i=1}^m A_i\Big)\Big| \leq \prod_{i=1}^m \operatorname{tr}(|A_i|)$.

Proof. Use mathematical induction. □

Proposition 2.8. *Let* $A, B \in M_n(\mathbb{C})$. *Then* $\operatorname{tr}(|A+B|) \leq \operatorname{tr}(|A|) + \operatorname{tr}(|B|)$. *In particular, for* $A_i, B_i \in M_n(\mathbb{C})$,

$$\operatorname{tr}\Big|\prod_{i=1}^{m} A_i + \prod_{i=1}^{m} B_i\Big| \leq \prod_{i=1}^{m} \operatorname{tr}\Big|A_i\Big| + \prod_{i=1}^{m} \operatorname{tr}\Big|B_i\Big|.$$

Proof. Using the fact that the trace of absolute matrix is equal to the sum of its singular values and the Fan singular value majorization theorem,

$$\operatorname{tr}(|A+B|) = \sum_{i=1}^{n} \sigma_i(A+B) \leq \sum_{i=1}^{n} \Big(\sigma_i(A) + \sigma_i(B)\Big) = \operatorname{tr}(|A|) + \operatorname{tr}(|B|).$$

Then corollary 2.7 implies that

$$\operatorname{tr}\left(\Big|\prod_{i=1}^{m} A_i + \prod_{i=1}^{m} B_i\Big|\right) \leq \prod_{i=1}^{m} \operatorname{tr}(|A_i|) + \prod_{i=1}^{m} \operatorname{tr}(|B_i|).$$

□

Theorem 2.9. *Let $A, C \in M_n(\mathbb{C})$ and let B, D be normal matrices. If $|A| \leq |C|$ and $|B| \leq |D|$, then* $\mathrm{tr}(|AB|^2) \leq \mathrm{tr}(|CD|^2)$.

Proof. Since $|A| \leq |C|$ and $|B| \leq |D|$, it follows that

$$A^*ABB^* \leq C^*CDD^*.$$

Since trace is a monotone function on the definite matrices and invariant under cyclic permutations, we get

$$\mathrm{tr}(|AB|^2) = \sum_{i=1}^{n} \lambda_i(ABB^*A^*) = \sum_{i=1}^{n} \lambda_i(A^*ABB^*) \leq \sum_{i=1}^{n} \lambda_i(C^*CDD^*) = \mathrm{tr}(|CD|^2).$$

□

Theorem 2.10. *Let $A, B \in M_n(\mathbb{C})$. Then* $\mathrm{tr}(|AB^*|^{2m}) \leq \mathrm{tr}\left(|A|^{2m}|B|^{2m}\right)$.

Proof. Since trace is invariant under cyclic permutations, it follows that

$$\mathrm{tr}(|AB^*|^{2m}) = \mathrm{tr}(BA^*AB^*)^m = \mathrm{tr}(A^*AB^*B)^m = \mathrm{tr}(|A|^2|B|^2)^m.$$

From ([2], Theorem 4), we have

$$\sum_{i=1}^{t} \lambda_i^m(AB) \leq \sum_{i=1}^{t} \lambda_i(A^m B^m), \qquad \leq t \leq n, \quad A, B \geq 0.$$

So we get $\sum_{i=1}^{t} \lambda_i^m(|A|^2|B|^2) \leq \sum_{i=1}^{t} \lambda_i\left(|A|^{2m}|B|^{2m}\right) = \mathrm{tr}\left(|A|^{2m}|B|^{2m}\right)$. This completes the proof. □

By Theorems 2.3 and 2.10, the following Corollary holds.

Corollary 2.11. *Let $A, B \in M_n(\mathbb{C})$. Then*
$\mathrm{tr}(|AB^*|^{2m}) \leq \left(\mathrm{tr}(|A|^{4m})\right)^{1/2}\left(\mathrm{tr}(|B|^{4m})\right)^{1/2}$.

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