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An application of Lyapunov functions to properties of solutions of a perturbed fractional differential system

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Abstract

This paper deals with a perturbed nonlinear system of fractional order differential equations (FrODEs) with Caputo derivative. The purpose of the paper is to discuss uniform stability (US), asymptotic stability (AS), Mittag-Leffer stability (MLS) of zero solution and boundedness at infinity of non-zero solutions of this perturbed nonlinear system of FrODEs with Caputo derivative. We obtain four new theorems on these mathematical concepts via a Lyapunov function (LF) and its Caputo derivative. For illustration, an example is provided which satisfies assumptions of the four new results and, in particular, shows their applications. The new results of this paper generalize and improve some recent ones in the literature and they have contributions to theory of FrODEs.

1 Introduction

As we know, calculations of integrals and derivatives of non-integer order are included in the theory of fractional calculus whose history dates back to

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the seventeenth century. Next, it can be seen from the relevant literature that the theory of fractional calculus is very popular in recent years and attracts the attention of numerous researchers due to its significant applications in various scientific fields such as engineering, control theory, physics, neural networks, population dynamics, ecology, medicine, applied mathematics (see, for example, [1], [7], [10], [17], [22], [23], [25], [46] and the references in these sources). Additionally, we know from the relevant literature that as the same integer order case in ODEs, one of the significant and attractive topic in qualitative theory of FrODEs is the investigation of the qualitative concepts of solutions of FrODEs such as US, AS, MLS, boundedness. In particular, for some results related to these concepts and some others qualitative concepts in ODEs and FrODEs, we refer the reader to [1-46] and the references of these scientific sources.
As for the motivation of this paper, it comes from the extensively current works in qualitative theory of FrODEs. Indeed, in recent years, we see that stability, US, AS, MLS, boundedness, convergence of solutions, etc., of numerous scalar FrODEs and systems of FrODEs with Caputo derivative, which are given in the form

$$ {}_{t_o}^{C}D_t^q x(t) = Ax(t) + f(t, x(t)), x(t_0) = x_0, \tag{1.1}$$

and numerous others forms of (1.1), have been studied and are still being investigated by many (see, for example, [2], [4], [6-13], [15], [17-20], [22-26], [39-46] and the references of these sources).

In view of this, we discuss the qualitative concepts of solutions of FrODEs with Caputo fractional derivative. We investigate a nonlinear perturbed system of FrODEs of Caputo type, which is given by

$$ {}_{t_o}^{C}D_t^q x(t) = -A(t)x(t) - f(t, x(t))x(t) + g(t, x(t)), x(t_0) = x_0, \tag{1.2}$$

where $x = (x_1, ..., x_n)^T \in \mathbb{R}^n, t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty), q \in (0, 1)$, $A(t) = (a_{ij}(t)) \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$, $A(t)$ is a symmetric matrix, $(i, j = 1, ..., n)$, $f = (f_1, ..., f_n)^T \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f_i(t, x(t)) = f_i(t, x_1(t), ..., x_n(t)), f(t, 0) = 0, g = (g_1, ..., g_n)^T \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g_i(t, x(t)) = g_i(t, x_1(t), ..., x_n(t))$ and $g(t, 0) = 0$. In this case, the nonlinear system of FrODEs (1.2) with Caputo derivative includes the zero solution.

As for the impact of our work, we would like to highlight the following:

1) To the best of our knowledge, the qualitative behaviors such as the stability, US, AS, MLS, boundedness, convergence of solutions, etc., of the nonlinear perturbed system of FrODEs (1.2) have not yet been investigated.

2) The main results of this paper, Theorems 3-6, and their conditions (A1)-(A4), which are given below, have very simple forms.

3) As a numerical application, we constructed an example, Example 1, satisfying conditions (A1)-(A4) of Theorems 3-6.

This paper is organized as follows: Section 2 gives some basic concepts and results related to the qualitative theory of FrODEs. In Section 3, we establish the basic assumptions of the main results of this paper and then four new theorems, Theorems 3-6, are presented and proved on the qualitative of behaviors of solutions. Section 4 gives an example, Example 1, as numerical applications of the main results of this paper. Finally, we conclude our paper in Section 6.

2 Basic concepts and results

In this section, some basic definitions, theorems and lemmas related to fractional calculus and stability analysis of a nonlinear system of FrODEs are given.

Consider a system of FrODEs with Caputo derivative:

$$ {}_{t_o}^{C}D_t^q x(t) = F(t, x(t)), \tag{2.3} $$

where $q \in (0,1)$, $F(t,0) = 0$, $F \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^+ = [0, \infty)$.

Definition 1. *(Duarte-Mermoud et al. [11, Definition 2]). A continuous function* $\gamma_1 : [0,t) \to [0,+\infty)$ *is said to belong to class* $-K$ *if it is strictly increasing and* $\gamma_1(0) = 0$.

Theorem 1. *(Duarte-Mermoud et al. [11, Theorem 3]). Let* $x = 0$ *be an equilibrium point for the non-autonomous system of FrODEs (3). Let us assume that there exist a continuous Lyapunov function* $V(t, x(t))$ *and a scalar class*$-K$ *function* $\gamma_1(.)$ *such that,* $\forall x(t) \neq 0$,

$$ \gamma_1(\|x(t)\|) \le V(t, x(t)) $$

and

$$ {}_{t_0}^{C}D_t^q V(t, x(t)) \le 0 \text{ with } q \in (0,1). $$

Then the origin of the system of FrODEs (2.3) is Lyapunov stable.

Moreover, if there is a scalar class $-K$ *function* $\gamma_2(.)$ *satisfying*

$$ V(t, x(t)) \le \gamma_2(\|x(t)\|), $$

then the origin of the system of FrODEs (2.3) is Lyapunov uniformly stable.

Theorem 2. *Duarte-Mermoud et al. [11, Theorem 1]). Let $x = 0$ be an equilibrium point for the non-autonomous system of FrODEs (2.3). Assume that there exists a Lyapunov function $V(t, x(t))$ and class $-K$ functions γ_i , $(i = 1, 2, 3)$, satisfying*

$$\gamma_1(\|x\|) \leq V(t,x) \leq \gamma_2(\|x\|),$$
$${}_{t_0}^{C}D_t^q V(t,x(t)) \leq -\gamma_3(\|x\|),$$

where $q \in (0,1)$. Then the system of FrODEs (2.3) is asymptotically stable.

Definition 2. *(Liu et al. [19, Definition 3.1]). The trivial solution of the system of FrODEs (2.3) is said to be Mittag-Leffler stable if*

$$\|x(t)\| \leq [m(x(t_0))E_v(-\sigma(t-t_0)^v)]^{\mu},$$

where $v \in (0,1]$,$\sigma \geq 0$, $\mu > 0$, $m(0) = 0$, $m(x) \geq 0$, $m(x)$ is locally Lipschitz with Lipschitz constant m_0, and

$$E_v(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk+1)}$$

is the one-parameter Mittag-Leffler function, and denotes the Gamma function.

Lemma 1. *([Liu et al. [19, Lemma 2.1]). Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable function. If a continuous function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ satisfies*

$${}_{t_o}^{C}D_t^q V(t,x(t)) \leq -\alpha V(t,x(t)),$$

then

$$V(t,x(t)) \leq V(t_0,x(t_0))E_q(-\alpha(t-t_0)^q),$$

where $\alpha > 0$ and $0 < q < 1$.

Lemma 2. *(Podlubny [25]).*

$${}_{t_o}^{C}D_t^q(ax(t)+by(t)) = a\,{}_{t_o}^{C}D_t^q x(t) + b\,{}_{t_o}^{C}D_t^q y(t),$$

where $q \in (0,1]$.

Lemma 3. *(Liu et al.[19, Lemma 2.2]). Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable function. Then, for any time instant $t \geq t_0$, the following relationship holds:*

$$\frac{1}{2}\,{}_{t_o}^{C}D_t^q(x^T Px) \leq x^T(t)P\,{}_{t_o}^{C}D_t^q x(t), \forall q \in (0,1],$$

where $P \in \mathbb{R}^{n\times n}$ is a constant, square, symmetric and positive definite matrix.

It is noteworthy that the nonlinear system of FrODEs (1.2) is included by the nonlinear system of FrODEs (2.3).Therefore, the above definitions, lemmas and theorems hold true for the nonlinear system of FrODEs (1.2).

The following lemma includes a well-known algebraic result.

Lemma 4. *(Graef et al [13, Lemma 2.8]). Let $x \in \mathbb{R}^n$, $n \in \mathbb{N}$ with $n \geq 1$, and $M \in \mathbb{R}^{n \times n}$ be a positive definite symmetric $n \times n-$ matrix such that*

$$\lambda_m \leq \lambda_i(M) \leq \lambda_M, (i = 1, 2, ..., n),$$

where $\lambda_i(M)$ denotes the eigenvalues of the matrix M. Then

$$\lambda_m \|x\|^2 \leq \langle Mx, x \rangle \geq \lambda_M \|x\|^2,$$

where λ_m, λ_M are the greatest and least eigenvalues of the matrix M, respectively.

It is known that λ_m and λ_M are positive since M is a positive definite symmetric matrix.

Let $x \in \mathbb{R}^n$. In this paper, the vector norm is defined by $\|x\| = \sum_{i=1}^{n} |x_i|$. We will use the definition of this norm.

3 Behaviors of solutions

The following conditions are essential and needed in our main results.

(A1) The matrices $A(t) \in C(\mathbb{R}^+, \mathbb{R}^{n \times n})$ and $f(t, x(t)) \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^{n \times n})$ are positive definite and symmetric and their eigenvalues satisfy

$$\lambda_i \left(A(t)\right) \geq \lambda_A > 0 \ \text{ for all } t \in \mathbb{R}^+,$$

$$\lambda_i \left(f(t, x)\right) \geq \lambda_f > 0 \ \text{ for all } t \in \mathbb{R}^+ \text{ and for all } x \in \mathbb{R}^n,$$

where λ_A, $\lambda_f \in \mathbb{R}$.

(A2) There is a positive constant g_M such that

$$g(t, 0) = 0, \|g(t, x(t))\| \leq g_M \|x(t)\| \quad \text{for all } t \in \mathbb{R}^+ \ \text{ and all } x \in \mathbb{R}^n.$$

(A3) There are positive constants λ_A, λ_f and g_M from (A1) and (A2), respectively, such that

$$\lambda_A + \lambda_f - g_M \geq 0.$$

(A4) There are positive constants $\lambda_A,\ \lambda_f$ and g_M from (A1) and (A2), respectively, and σ_0 such that

$$\lambda_A + \lambda_f - g_M \ \geq \sigma_0.$$

Theorem 3. *The zero solution of the system of FrODEs (1.2) is uniformly stable if conditions (A1)-(A3) are satisfied.*

Proof.
We define a LF $\Delta := \Delta(t, x(t))$ by

$$\Delta(t, x(t))) := \langle x(t), x(t)\rangle = \|x(t)\|^2 = \sum_{i=1}^{n} x_i^2(t). \tag{3.4}$$

From this point, it is clear that the LF $\Delta(t, x(t))$ in (3.4) satisfies the following relations:

$$\Delta(t, 0) = 0,$$

$$k_1 \sum_{i=1}^{n} x_i^2(t) = k_1\|x(t)\|^2 \leq \Delta(t, x(t)),$$

and

$$\Delta(t, x(t)) \leq k_2 \sum_{i=1}^{n} x_i^2(t) = k_2\|x(t)\|^2;$$

i.e.,

$$k_1\|x(t)\|^2 \leq \Delta(t, x(t)) \leq k_2\|x(t)\|^2,$$

where $k_1 \in (0, 1]$ and $k_2 \geq 1$.

Taking the derivative of the LF $\Delta(t, x(t))$ in (3.4) along solutions of the system of FrODEs (1.2), using conditions (A1), (A2) and some elementary calculations, we get

$$\begin{aligned}
{}_{t_o}^{C}D_t^q \Delta(t, x(t)) &\leq 2\left\langle x(t), {}_{t_o}^{C}D_t^q x(t)\right\rangle \\
&= -2\langle x(t), A(t)x(t)\rangle - 2\langle x(t), f(t, x(t))x(t)\rangle + 2\langle x(t), g(x(t))\rangle \\
&\leq -2\lambda_a \langle x(t), x(t)\rangle - 2\lambda_f \langle x(t), x(t)\rangle + 2\|x(t)\| \ \ \|g(t, x(t))\| \\
&\leq -2\lambda_a \langle x(t), x(t)\rangle - 2\lambda_f \langle x(t), x(t)\rangle + 2g_M\|x(t)\|^2 \\
&= -2\left[\lambda_a + \lambda_f - g_M\right]\|x(t)\|^2 \ \leq 0;
\end{aligned}$$

i.e., we have

$${}_{t_o}^{C}D_t^q \Delta(t, x(t)) \leq 0.$$

Then the above mathematical evaluations implies that zero solution of the system of FrODEs (1.2) is uniformly stable. The proof of Theorem 3 is complete.

In the following result, we investigate the boundedness of solutions of the system of FrODEs (1.2).

Theorem 4. *The solutions of the system of FrODEs (1.2) are bounded at the infinity if conditions (A1)-(A3) are satisfied.*

Proof.
By conditions (A1)-(A3), we have

$$ {}_{t_o}^{C}D_t^q \Delta(t, x(t)) \leq 0. $$

This result verifies that the LF $\Delta(t, x(t))$ is decreasing. Because of this fact, as for the next step, it follows that

$$ \Delta(t, x(t)) \leq \Delta(t_0, x(t_0)),\ t \geq t_0. $$

From the definition of the LF of (3.4), as for the next step, we derive

$$ \|x(t)\|^2 = \sum_{i=1}^{n} x_i^2(t) = \Delta(t, x(t)) \leq \Delta(t_0, x(t_0)) = \sum_{i=1}^{n} x_i^2(t_0). $$

This result implies that

$$ \|x(t)\| \leq \sqrt{\sum_{i=1}^{n} x_i^2(t_0)} = B_0 > 0 \textit{ provided that } x(t_0) \neq 0. $$

Hence, it follows that $\|x(t)\| \leq B_0$ as $t \to \infty$. This finishes the proof of Theorem 4.

In the next theorem, we discuss the asymptotic stability of the system of FrODEs (1.2).

Theorem 5. *The zero solution of the system of FrODEs (1.2) is asymptotically stable if conditions (A1), (A2), and (A4) are satisfied.*

Proof.
In view of the above mathematical calculations and conditions (A1), (A2), and (A4), we get

$$ {}_{t_o}^{C}D_t^q \Delta(t, x(t)) \leq -2\sigma_0 \|x(t)\|^2 < 0, $$

provided that $\|x(t)\| \neq 0$. This result and the discussion of Theorem 3 show that the zero solution of the system of FrODEs (1.2) is asymptotically stable. This finishes the proof of Theorem 5.

The following theorem, Theorem 6, studies the Mittag-Leffler stability of the zero solution of the system of FrODEs (1.2).

Theorem 6. *The zero solution of the system of FrODEs (1.2) is Mittag-Leffler stable if conditions (A1), (A2) and (A4) are satisfied.*

Proof.
By the conditions of Theorem 6, it follows that

$$ {}_{t_o}^{C}D_t^q \Delta(t, x(t)) \leq -2\sigma_0 \|x(t)\| = -2\sigma_0 \Delta(t, x(t)). $$

Indeed, we get

$$ {}_{t_o}^{C}D_t^q \Delta(t, x(t)) \leq -2\sigma_0 \Delta(t, x(t)). $$

Using Lemma 1, we can write

$$ \begin{aligned} \|x(t)\|^2 &= \Delta(t, x(t)) \leq \Delta(t_0, x(t_0)) E_q(-2\sigma_0 (t - t_0)^q) \\ &= \|x(t_0)\|^2 E_q(-2\sigma_o (t - t_0)^q) \\ &= [m(x(t_0)) E_q(-2\sigma_0 (t - t_0)^q)], \end{aligned} $$

where $m(x_0) = \|x(t_0)\|^2$. It is also followed that $m(x) = \|x(t)\|^2$, $m(0) = 0$, $m(x) \geq 0$, $m(x)$ satisfies the locally Lipschitz condition. Next, we have

$$ \|x(t)\| \leq \sqrt{[m(x(t_0)) E_q(-2\sigma_0 (t - t_0)^q)]}. $$

Thus, the proof of Theorem 6 is completed by using Definition 2.

4 Numerical Applications

Example 7. *Consider the following system of FrODEs with Caputo derivative, $q \in (0, 1)$:*

$$ \begin{aligned} \begin{pmatrix} {}_{t_o}^{C}D_t^q x_1(t) \\ {}_{t_o}^{C}D_t^q x_2(t) \end{pmatrix} = &- \begin{bmatrix} 24 + \frac{1}{1+\exp(t^2)} & 1 \\ 1 & 24 + \frac{1}{1+\exp(t^2)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &- \begin{bmatrix} 18 + \frac{1}{1+\exp(t^2)+|x_1(t)|} & 2 \\ 2 & 18 + \frac{1}{1+\exp(t^2)+|x_2(t)|} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} \frac{2x_1(t)}{4+\exp(t^2)+|x_1(t)|} \\ \frac{2x_2(t)}{4+\exp(t^2)+|x_2(t)|} \end{bmatrix} \end{aligned} \tag{4.5} $$

Comparing the systems of FrODEs (4.5) and (1.2), we obtain the below relations:

$$A(t) = \begin{bmatrix} 24 + \frac{1}{1+\exp(t^2)} & 1 \\ 1 & 24 + \frac{1}{1+\exp(t^2)} \end{bmatrix},$$

$$f(t, x(t)) = \begin{bmatrix} 18 + \frac{1}{1+\exp(t^2)+|x_1(t)|} & 2 \\ 2 & 18 + \frac{1}{1+\exp(t^2)+|x_2(t)|} \end{bmatrix},$$

$$g(t, x(t)) = \begin{bmatrix} \dfrac{2x_1(t)}{4 + \exp(t^2) + |x_1(t)|} \\ \dfrac{2x_2(t)}{4 + \exp(t^2) + |x_2(t)|} \end{bmatrix}.$$

Then, by means of some elementary calculations, we obtain the following relations:

$$\lambda_1(A(t)) = 23 + \frac{1}{1 + \exp(t^2)},$$

$$\lambda_2(A(t)) = 25 + \frac{1}{1 + \exp(t^2)},$$

$$23 = \lambda_A \leq \lambda_i(A(t)),$$

$$\lambda_1(f(t, x(t))) = 16 + \frac{1}{1 + \exp(t^2) + |x_1(t)|},$$

$$\lambda_2(f(t, x(t))) = 20 + \frac{1}{1 + \exp(t^2) + |x_2(t)|},$$

$$16 = \lambda_f \leq \lambda_i(f(t, x(t))),$$

$$g(t, x(t)) = \begin{bmatrix} g_1(t, x_1(t), x_2(t)) \\ g_2(t, x_1(t), x_2(t)) \end{bmatrix} = \begin{bmatrix} \dfrac{2x_1(t)}{1 + \exp(t^2) + |x_1(t)|} \\ \dfrac{2x_2(t)}{1 + \exp(t^2) + |x_2(t)|} \end{bmatrix},$$

$$g(t, 0) = 0,$$

$$\|g(t, x(t))\| = \left\| \begin{bmatrix} g_1(t, x_1(t), x_2(t)) \\ g_2(t, x_1(t), x_2(t)) \end{bmatrix} \right\| = \left\| \begin{bmatrix} \dfrac{2x_1(t)}{1 + \exp(t^2) + |x_1(t)|} \\ \dfrac{2x_2(t)}{1 + \exp(t^2) + |x_2(t)|} \end{bmatrix} \right\|$$

$$\leq 2\,|x_1| + 2\,|x_2| = 2\,\|x\|,$$

$$g_M = 2,$$

$$\lambda_A + \lambda_f - g_M = 23 + 16 - 2 = 37 > 36 = \sigma_0 > 0.$$

These relations demonstrate that conditions (A1)-(A4) hold. This means that the conditions of Theorems 3-4 can be applied and verified. For these reasons, one can say that the zero solution of the nonlinear system of FrODEs (4.5) is uniformly stable, asymptotically stable and Mittag-Leffler stable and the nonzero solutions of FrODEs (4.5) are bounded at infinity.

5 Conclusion

In this paper, we considered a modified nonlinear perturbed system of FrODEs with Caputo derivative. We presented four new qualitative results related to the US, AS, MLS and boundedness of solutions of the modified perturbed system of FrODEs with Caputo derivative. To demonstrate these results, we used a LF as a basic tool via the Lyapunov direct method and we showed its application in fractional calculus. Our results are more general and suitable for applications in fractional calculus. We provided a two dimensional example as a numerical application to demonstrate the applicability of the main results of this paper.

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