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The Relationship of Modular Lattice with Maximum Pre-period Property and Minimum Pre-period Property

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Abstract

The pre-period of a lattice (resp. pre-period of a lattice fixing α) $\mathbf{A}$ is defined as the supremum of the pre-period of an endomorphism (resp. a connected endomorphism fixing α) of $\mathbf{A}$ and is denoted by $\lambda(\mathbf{A})$ (resp. $\lambda_\alpha(\mathbf{A})$). Let $\mathcal{MPP}$ (resp. $\mathcal{NPP}$) be the set of all finite modular lattices $\mathbf{A}$ with $\lambda(\mathbf{A}) = \ell(\mathbf{A})$ (resp. $\lambda(\mathbf{A}) = \max\{\lambda_0(\mathbf{A}), \lambda_1(\mathbf{A})\}$,) where $\ell(\mathbf{A})$ is the length of $\mathbf{A}$. We will show that $\mathcal{MPP} \subsetneq \mathcal{NPP} \subsetneq \mathcal{M}$, where $\mathcal{M}$ is the set of all finite modular lattices.

1 Introduction

One of the most important tools in studying universal algebra is the notion of endomorphism. An endomorphism f of a structure A can be con-

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sidered as a unary operation and $\langle A; f\rangle$ is a *monounary algebra*. Some properties of monounary algebras connected with the notion of homomorphism were studied (for instance, in [4], [7], [9]).

The importance of the theory of unary and monounary algebras was pointed out for example in the monographs [6] and [8]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. If the graph of a monounary algebra is connected, then it is called a *connected monounary algebra*. As every graph is a sum of connected components, every monounary algebra is a sum of connected monounary algebras.

Let $f : A \to A$ be a unary operation on a set A. Let f^0 be the identity map on A and $\mathrm{Im}(f) := \{f(a) \mid a \in A\}$. A *pre-period* (or *stabilizer*) of f is the least nonnegative integer n satisfying $\mathrm{Im} f^n = \mathrm{Im} f^{n+1}$ and denoted by $\lambda(f)$ [12]. An operation f on A is *connected* if for each $a, b \in A$, there exist nonnegative integers n and m such that $f^n(a) = f^m(b)$. Some results from [3] and [10] imply that $\lambda(f) \leq |A| - 1$ and they characterize f with $\lambda(f) = |A| - 1$. Moreover, if $\lambda(f) = |A| - 1$, then f is connected.

Several authors have focused, in particular, on connected monounary algebras (see [11], [5], for example). If f is a connected order-preserving map on a bounded poset $\mathbf{P}$, then f has a unique fixed point α [1]. Moreover, $\lambda(f) \leq \ell(\mathbf{P})$, where $\ell(\mathbf{P})$ stands for the length of $\mathbf{P}$. The *pre-period* of a finite lattice $\mathbf{A}$ *fixing* α is the maximum of $\lambda(f)$ whose f is a connected endomorphism on $\mathbf{A}$ and α is the fixed point and it is denoted by $\lambda_\alpha(\mathbf{A})$ which was studied in the case $\alpha = 0$. They showed that if $\mathbf{A}$ is distributive, then $\lambda_0(\mathbf{A}) \leq \ell(\mathbf{A})$ and they characterized $\mathbf{A}$ with $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$.

These facts were generalized in [2] to an endomorphism without the connectivity. The supremum of the pre-period of endomorphism of a lattice $\mathbf{A}$ is called the *pre-period* of $\mathbf{A}$, denoted by $\lambda(\mathbf{A})$, which is less than or equal to the length of $\mathbf{A}$ if it is finite modular. Let $\mathcal{M}$ be the set of all finite modular lattices. A lattice $\mathbf{A} \in \mathcal{M}$ is said to have the maximum pre-period property (briefly MPP) if $\lambda(\mathbf{A}) = \ell(\mathbf{A})$ and we denote the set of all modular lattices having the MPP by $\mathcal{MPP}$.

By the definitions, one can see that $\max\{\lambda_0(\mathbf{A}), \lambda_1(\mathbf{A})\} \leq \lambda(\mathbf{A})$. A lattice $\mathbf{A} \in \mathcal{M}$ is said to have the minimum pre-period property (briefly NPP) if $\max\{\lambda_0(\mathbf{A}), \lambda_1(\mathbf{A})\} = \lambda(\mathbf{A})$ and we denote the set of all modular lattices having the NPP by $\mathcal{NPP}$. In this paper, we will show that $\mathcal{MPP} \subsetneq \mathcal{NPP} \subsetneq \mathcal{M}$ by determining the pre-period of the glued sum of Boolean with a lattice.

2 Preliminaries

For a finite lattice $\mathbf{A}$, one can see that there exists the top 1 and the bottom 0. Moreover, for an endomorphism f on $\mathbf{A}$, f is connected fixing 0 if and only if $f^n(1) = 0$, for some non-negative integer n. This implies that the pre-period $\lambda(f)$ of a connected endomorphism f on a finite lattice $\mathbf{A}$ fixing 0 is the least non-negative integer with $f^{\lambda(f)}(1) = 0$. In [1], for a finite distributive lattice $\mathbf{A}$ It was shown that $\lambda_0(\mathbf{A}) \leq \ell(\mathbf{A})$. A condition of $\mathbf{A}$ with $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ is given in the following theorem which can be stated for any finite modular lattice.

Theorem 2.1. *[1] Let $\mathbf{A}$ be a finite modular lattice. Then $\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ if and only if there is an endomorphism f on $\mathbf{A}$ such that $0 = f^{\lambda(\mathbf{A})}(1) \prec f^{\lambda(\mathbf{A})-1}(1) \prec \ldots \prec f(1) \prec 1$. Moreover, f is connected.*

One of the simple lattices satisfying the condition in Theorem 2.1 is a finite chain or a finite directed product of the 2-element chain. Indeed, for an m-element chain $\mathbf{C}_m = \{c_1 \prec c_2 \prec \ldots \prec c_m\}$, the function $f : C_m \to C_m$ defined by

$$f(c_i) = \begin{cases} c_{i-1} & \text{if } i > 1, \\ c_1 & \text{if } i = 1 \end{cases}$$

is an endomorphism on $\mathbf{C}_m$ with

$$c_1 = f^{m-1}(c_m) \prec c_2 = f^{m-2}(c_m) \prec \ldots \prec c_{m-1} = f(c_m) \prec c_m.$$

So $\lambda(\mathbf{C}_m) = |\mathbf{C}_m| - 1 = m - 1$.

Theorem 2.2. *[1] The pre-period of the directed product $\mathbf{2}^n$ of the 2-element chain $\mathbf{2}$ is equal to n, for all $n \in \mathbb{N}$; that is, $\lambda_0(\mathbf{2}^n) = n$.*

Lemma 2.3. *[2] Let $\mathbf{L}^\partial$ be the dual of a bounded lattice $\mathbf{L}$. Then $\lambda_0(\mathbf{L}) = \lambda_1(\mathbf{L}^\partial)$.*

Theorem 2.4. *[2] Let $\mathbf{L}$ be a finite modular lattice. Then*

$$\lambda_\alpha(\mathbf{L}) \leq \lambda(\mathbf{L}) \leq \ell(\mathbf{L})$$

, for all $\alpha \in L$.

Theorem 2.5. *[2] Let $\mathbf{L}$ be a finite modular lattice. Then*

$\mathbf{L}$ has the MPP if and only if either $\lambda_0(\mathbf{L}) = \ell(\mathbf{L})$ or $\lambda_1(\mathbf{L}) = \ell(\mathbf{L})$.

Corollary 2.6. *$\mathcal{MPP} \subseteq \mathcal{NPP}$.*

Let $\mathbf{A}$ and $\mathbf{B}$ be (disjoint) ordered sets. The *linear sum* $\mathbf{A} \oplus \mathbf{B}$ is defined by taking the following order relation on $A \cup B$: $a \leq b$ if and only if $a \leq b$ in $\mathbf{A}$ or in $\mathbf{B}$ or $(a, b) \in A \times B$. If $\mathbf{A}$ has the top $1_{\mathbf{A}}$ and $\mathbf{B}$ has the bottom $0_{\mathbf{B}}$, the *glued sum* of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \dotplus \mathbf{B}$, is obtained from the linear sum by identifying $1_{\mathbf{A}}$ with $0_{\mathbf{B}}$.

Theorem 2.7. *[2] Let $\mathbf{A}$ and $\mathbf{B}$ be non-trivial finite modular lattices. Then $\mathbf{A} \dotplus \mathbf{B}$ has the MPP if and only if either*

1. *$\lambda_0(\mathbf{A}) = \ell(\mathbf{A})$ and $\mathbf{B}$ is a chain, or*
2. *$\lambda_1(\mathbf{B}) = \ell(\mathbf{B})$ and $\mathbf{A}$ is a chain.*

Theorem 2.8. *[2] Let $\mathbf{A}$ and $\mathbf{B}$ be finite lattices. Then*

$$\lambda_i(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda_i(\mathbf{A}) + \lambda_i(\mathbf{B})$$

and

$$\lambda(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda(\mathbf{A}) + \lambda(\mathbf{B}),$$

for $i \in \{0, 1\}$.

3 A pre-period of the Glued sum of Boolean

We will start this section by finding some formulas of the pre-period of glued sum of lattices.

Theorem 3.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite non-trivial lattices.*

1. *If $\alpha \in A$, then*
$$\lambda_\alpha(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda_\alpha(\mathbf{A}) + \lambda_0(\mathbf{B}).$$
2. *If $\alpha \in B$, then*
$$\lambda_\alpha(\mathbf{A} \dotplus \mathbf{B}) \leq \lambda_1(\mathbf{A}) + \lambda_\alpha(\mathbf{B}).$$

Proof. Let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$ and let f be a connected endomorphism on $\mathbf{D}$ fixing $\alpha \in A$. It is clear that $f \restriction_A$ and the map $g : B \to B$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in B, \\ 0_{\mathbf{B}} & \text{if } f(x) \in A \end{cases}$$

are connected endomorphisms on $\mathbf{A}$ fixing α and on $\mathbf{B}$ fixing $0_{\mathbf{B}}$, respectively. Suppose that $\lambda(f\downharpoonleft_A) = m$ and $\lambda(g) = n$. Then $f^n(1_{\mathbf{B}}) \in A$ and

$$f^{m+n}(D) = f^n(f^m(A)) \cup f^m(f^n(B)) = \{\alpha\}.$$

Hence, $\lambda(f) \leq m+n \leq \lambda_\alpha(\mathbf{A})+\lambda_0(\mathbf{B})$; and so, $\lambda_\alpha(\mathbf{A}\dotplus\mathbf{B}) \leq \lambda_\alpha(\mathbf{A})+\lambda_0(\mathbf{B})$. Similarly, if $\alpha \in B$, then $\lambda_\alpha(\mathbf{A}\dotplus\mathbf{B}) \leq \lambda_1(\mathbf{A}) + \lambda_\alpha(\mathbf{B})$. ☐

Proposition 3.2. *For finite non-trivial lattices* $\mathbf{A}$ *and* $\mathbf{B}$*,*

$$\lambda(\mathbf{A}\dotplus\mathbf{B}) \geq \max\{\lambda(\mathbf{A}), \lambda(\mathbf{B})\} \ \textit{and}$$

$$\lambda_\alpha(\mathbf{A}\dotplus\mathbf{B}) \geq \lambda_\alpha(\mathbf{A})\,(\textit{resp. } \lambda_\alpha(\mathbf{B}))$$

for all $\alpha \in \mathbf{A}$ *(resp.* $\alpha \in \mathbf{B}$*).*

Proof. Let $\mathbf{D} = \mathbf{A} \dotplus \mathbf{B}$. Let $f : \mathbf{A} \to \mathbf{A}$ be an endomorphism. Then the map $g : D \to D$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ f(1_{\mathbf{A}}) & \text{if } x \notin A \end{cases}$$

is an endomorphism on $\mathbf{D}$ with $\lambda(f) = \lambda(g)$. Moreover, if f is connected fixing α, then so is g. So, $\lambda(\mathbf{A}) \leq \lambda(\mathbf{D})$ and $\lambda_\alpha(\mathbf{A}) \leq \lambda_\alpha(\mathbf{D})$. ☐

Lemma 3.3. *Let* $\mathbf{A}$ *be a finite non-trivial lattice and* $f : \mathbf{A}\dotplus\mathbf{2}^n \to \mathbf{A}\dotplus\mathbf{2}^n$ *be an endomorphism.*

1. *If* $f(1_{\mathbf{A}}) = 1_{\mathbf{A}}$*, then* $\lambda(f) \leq \max\{\lambda(\mathbf{A}), n\}$*. Moreover, if* f *is connected, then* $\lambda(f) \leq \max\{\lambda_1(\mathbf{A}), n\}$*.*
2. *If* $f(1_{\mathbf{A}}) < 1_{\mathbf{A}}$*, then* $\lambda(f) \leq \lambda(\mathbf{A}) + 1$*. Furthermore, if* f *is connected with* $f(\alpha) = \alpha$*, then* $\lambda(f) \leq \lambda_\alpha(\mathbf{A}) + 1$*.*

Proof. (1) Suppose that $f(1_{\mathbf{A}}) = 1_{\mathbf{A}}$ and $M := \max\{\lambda(\mathbf{A}), n\}$. Then $f\downharpoonleft_A$ and $f\downharpoonleft_{\mathbf{2}^n}$ are endomorphisms on $\mathbf{A}$ and $\mathbf{2}^n$, respectively. So,

$$\begin{aligned} f^M(\mathbf{A}\dotplus\mathbf{2}^n) &= f\downharpoonleft_A^M(\mathbf{A}) \cup f\downharpoonleft_{\mathbf{2}^n}^M(\mathbf{2}^n) \\ &= f\downharpoonleft_A^{M+1}(\mathbf{A}) \cup f\downharpoonleft_{\mathbf{2}^n}^{M+1}(\mathbf{2}^n) = f^{M+1}(\mathbf{A}\dotplus\mathbf{2}^n). \end{aligned}$$

Hence, $\lambda(f) \leq M$. Similarly, if f is connected, then $\lambda(f) \leq \max\{\lambda_1(\mathbf{A}), n\}$.
(2) For convenient, let $\mathbf{B} = \mathbf{2}^n$ and $a_i = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)$ be an atom of $\mathbf{B}$. Suppose that $f(1_{\mathbf{A}}) < 1_{\mathbf{A}}$; that is, $f(0_{\mathbf{B}}) < 0_{\mathbf{B}}$. Then for each $i \neq j$,

$f(a_i) \wedge f(a_j) = f(0_{\mathbf{B}}) < 0_{\mathbf{B}}$; and so, $f(a_i) < 0_{\mathbf{B}}$ or $f(a_j) < 0_{\mathbf{B}}$. We may assume that $f(a_1), \ldots, f(a_{n-1}) < 0_{\mathbf{B}}$.
case (i): $f(a_n) < 0_{\mathbf{B}}$. Then $f(a_1), \ldots, f(a_n) \in A$ which implies that $f(\mathbf{A} \dot{+} \mathbf{B}) \subseteq \mathbf{A}$; and so

$$f^{\lambda(\mathbf{A})+1}(\mathbf{A}) \subseteq f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dot{+} \mathbf{B}) = f^{\lambda(\mathbf{A})}(f(\mathbf{A} \dot{+} \mathbf{B})) \subseteq f^{\lambda(\mathbf{A})}(\mathbf{A}) = f^{\lambda(\mathbf{A})+1}(\mathbf{A}).$$

Thus,

$$f^{\lambda(\mathbf{A})+2}(\mathbf{A} \dot{+} \mathbf{B}) = f^{\lambda(\mathbf{A})+2}(\mathbf{A}) = f^{\lambda(\mathbf{A})+1}(\mathbf{A}) = f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dot{+} \mathbf{B}).$$

So, $\lambda(f) \leq \lambda(\mathbf{A}) + 1$.
case (ii): $f(a_n) \geq 0_{\mathbf{B}}$. Then

$$f(1_{\mathbf{B}}) = f(a_1 \vee, \ldots \vee a_n) = f(a_1) \vee \ldots \vee f(a_n) = f(a_n).$$

Since

$$f(a_n) = f(0, \ldots, 0, 1) \leq f(x_1, \ldots, x_{n-1}, 1) \leq f(1_{\mathbf{B}}),$$

we get

$$f(x_1, \ldots, x_{n-1}, 1) = f(1_{\mathbf{B}}) \tag{3.1}$$

for all $x_i \in \{0, 1\}$ and $1 \leq i \leq n$. For each $i \in \{0, 1\}$, let B_i be the set of elements $(x_1, \ldots, x_n) \in B$ with $x_n = i$. Then $f(B_0) \subseteq A$ and $f(B_1) = \{f(1_{\mathbf{B}})\}$. Since

$$\begin{aligned} f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dot{+} \mathbf{B}) &= f^{\lambda(\mathbf{A})}(f(A) \cup f(B_0 \cup B_1)) \\ &\subseteq f^{\lambda(\mathbf{A})}(f(A)) \cup f^{\lambda(\mathbf{A})}(A) \cup f^{\lambda(\mathbf{A})+1}(B_1) \\ &= f^{\lambda(\mathbf{A})+1}(A) \cup f^{\lambda(\mathbf{A})+1}(A) \cup f^{\lambda(\mathbf{A})+1}(B_1) \\ &\subseteq f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dot{+} \mathbf{B}), \end{aligned}$$

we get $f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dot{+} \mathbf{B}) = f^{\lambda(\mathbf{A})+1}(A) \cup f^{\lambda(\mathbf{A})+1}(B_1)$. If $f(1_{\mathbf{B}}) \in B_1$, then $f^2(B_1) = f(B_1)$; and so,

$$\begin{aligned} f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dot{+} \mathbf{B}) &= f^{\lambda(\mathbf{A})+1}(A) \cup f^{\lambda(\mathbf{A})+1}(B_1) \\ &= f^{\lambda(\mathbf{A})+2}(A) \cup f^{\lambda(\mathbf{A})+2}(B_1) \\ &= f^{\lambda(\mathbf{A})+2}(\mathbf{A} \dot{+} \mathbf{B}). \end{aligned}$$

Suppose that $f(1_{\mathbf{B}}) = x = (x_1, \ldots, x_{n-1}, 0)$ and x' be the complement of x in $\mathbf{B}$. By (3.1), $f(x') = f(1_{\mathbf{B}})$ and

$$f^2(1_{\mathbf{B}}) = f(x) = f(x) \wedge f(1_{\mathbf{B}}) = f(x) \wedge f(x') = f(0_{\mathbf{B}}) = f(1_{\mathbf{A}})$$

which implies that $f^2(B_1) = \{f(1_{\mathbf{A}})\}$. Since $\mathbf{A}$ is non-trivial, we get $\lambda(A) \geq 1$ and

$$\begin{aligned} f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dotplus \mathbf{B}) &= f^{\lambda(\mathbf{A})+1}(A) \cup f^{\lambda(\mathbf{A})-1}(f^2(B_1)) \\ &= f^{\lambda(\mathbf{A})+1}(A) \cup f^{\lambda(\mathbf{A})-1}(\{f(1_{\mathbf{A}})\}) \\ &= f^{\lambda(\mathbf{A})}(A) \cup f^{\lambda(\mathbf{A})}(\{1_{\mathbf{A}}\}) \\ &= f^{\lambda(\mathbf{A})}(A). \end{aligned}$$

So, $f^{\lambda(\mathbf{A})+1}(\mathbf{A} \dotplus \mathbf{B}) = f^{\lambda(\mathbf{A})+2}(\mathbf{A} \dotplus \mathbf{B})$ implies that $\lambda(f) \leq \lambda(\mathbf{A}) + 1$. Note that if f is connected with $f(\alpha) = \alpha$, then $\lambda(f) \leq \lambda_\alpha(\mathbf{A}) + 1$. □

The next corollary follows from Proposition 3.2 and Lemma 3.3.

Corollary 3.4. *Let α be an element in a finite non-trivial lattice $\mathbf{A}$.*

1. *If $\alpha = 1_{\mathbf{A}}$, then $\lambda_\alpha(\mathbf{A} \dotplus \mathbf{2}^n) = \max\{\lambda_\alpha(\mathbf{A}), n\}$.*
2. *If $\alpha \neq 1_{\mathbf{A}}$, then*

$$\lambda_\alpha(\mathbf{A} \dotplus \mathbf{2}^n) \text{ is either } \lambda_\alpha(\mathbf{A}) \text{ or } \lambda_\alpha(\mathbf{A}) + 1.$$

Theorem 3.5. *The pre-period of the glued sum $\mathbf{2}^m \dotplus \mathbf{2}^n$ is $\max\{m+1, n+1\}$. Moreover, $\lambda_0(\mathbf{A} \dotplus \mathbf{2}^n) = \lambda_0(\mathbf{A}) + 1$, for all finite non-trivial lattices $\mathbf{A}$.*

Proof. Lemma 3.3 implies that $\lambda(\mathbf{2}^m \dotplus \mathbf{2}^n) \leq \max\{m+1, n+1\}$. Let $\mathbf{A}$ be a finite non-trivial lattice and f be a connected endomorphism on $\mathbf{A}$ with $f(0) = 0$ and $\lambda(f) = \lambda_0(\mathbf{A})$. Define $g : \mathbf{A} \dotplus \mathbf{2}^n \to \mathbf{A} \dotplus \mathbf{2}^n$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 1_{\mathbf{A}} & \text{if } x = (x_1, x_2, \ldots, x_{n-1}, 1) \in B, \\ f(1_{\mathbf{A}}) & \text{if } x = (x_1, x_2, \ldots, x_{n-1}, 0) \in B. \end{cases}$$

Then g is a connected endomorphism on $\mathbf{A} \dotplus \mathbf{2}^n$ fixing $0_{\mathbf{A}}$. So, $\lambda(g) \geq \lambda(f) + 1 = \lambda_0(\mathbf{A}) + 1$ and $\lambda_0(\mathbf{A} \dotplus \mathbf{2}^n) \geq \lambda_0(\mathbf{A}) + 1$.

If $\mathbf{A} = \mathbf{2}^m$, then Theorem 2.2 and Lemma 2.3 imply that $\lambda(\mathbf{2}^m \dotplus \mathbf{2}^n) \geq \lambda_0(\mathbf{2}^m \dotplus \mathbf{2}^n) \geq m+1$ and $\lambda(\mathbf{2}^m \dotplus \mathbf{2}^n) \geq \lambda_1(\mathbf{2}^m \dotplus \mathbf{2}^n) = \lambda_0(\mathbf{2}^n \dotplus \mathbf{2}^m) \geq n+1$. □

Corollary 3.6. *$\mathcal{MPP} \subsetneq \mathcal{NPP}$. Moreover, the glued sum of Boolean lattices has the NPP.*

Proof. By Theorem 2.7, $\mathbf{2}^m \dotplus \mathbf{2}^n$ has no the MPP for $m, n \geq 2$. Moreover, by Theorem 3.5, $\mathbf{2}^m \dotplus \mathbf{2}^n$ has the NPP. □

Theorem 3.7. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite non-trivial lattices. Then*

$$\lambda_0(\mathbf{A} \oplus \mathbf{B}) = \lambda_0(\mathbf{A}) + \lambda_0(\mathbf{2} \dot{+} \mathbf{B}) \text{ and}$$

$$\lambda_1(\mathbf{A} \oplus \mathbf{B}) = \lambda_1(\mathbf{A} \dot{+} \mathbf{2}) + \lambda_1(\mathbf{B}).$$

Proof. Theorem 3.1 implies that $\lambda_0(\mathbf{A} \oplus \mathbf{B}) \leq \lambda_0(\mathbf{A}) + \lambda_0(\mathbf{2} \dot{+} \mathbf{B})$. Let $\mathbf{D} = \mathbf{A} \oplus \mathbf{B}$ and $\mathbf{B}' = \mathbf{2} \dot{+} \mathbf{B}$. Let $f_A : A \to A$ and $f_{B'} : \mathbf{B}' \to \mathbf{B}'$ be connected endomorphisms on $\mathbf{A}$ and $\mathbf{B}'$ fixing $0_{\mathbf{A}}$ and $0_{\mathbf{B}'}$, respectively. Define $f_D : D \to D$ by

$$f_D(x) = \begin{cases} f_A(x) & \text{if } x \leq 1_{\mathbf{A}}, \\ f_{B'}(x) & \text{if } x > 1_{\mathbf{A}}. \end{cases}$$

It is easy to see that f_D is a connected endomorphism on $\mathbf{D}$ with $f_D(0_{\mathbf{D}}) = 0_{\mathbf{D}}$.

Suppose that $\lambda(f_A) = n$ and $\lambda(f_{B'}) = m$. Then

$$f_D^{m+n-1}(1_{\mathbf{D}}) = f_D^{n-1}(f_{B'}^{m}(1_{\mathbf{D}})) = f_D^{n-1}(0_{\mathbf{B}'}) = f_A^{n-1}(1_{\mathbf{A}}) > 0_{\mathbf{D}}$$

which implies that $m+n \leq \lambda(f_D) \leq \lambda_0(\mathbf{D})$. So, $\lambda_0(\mathbf{A})+\lambda_0(\mathbf{B}') \leq \lambda_0(\mathbf{D})$. □

Corollary 3.8. *For each non-trivial lattice* $\mathbf{A}$ *and natural number* n*,*

$$\lambda_0(\mathbf{A} \oplus \mathbf{2}^n) = \lambda_0(\mathbf{A}) + 2.$$

Theorem 3.9. $\mathcal{NPP} \subsetneq \mathcal{M}$.

Proof. Let $\mathbf{L} = \mathbf{2} \dot{+} \mathbf{2}^5 \dot{+} \mathbf{2} \dot{+} \mathbf{2}^5 \dot{+} \mathbf{2}$. Theorems 3.5 and 3.7 imply that

$$\lambda_0(\mathbf{L}) = \lambda_0(\mathbf{2} \dot{+} \mathbf{2}^5) + \lambda_0(\mathbf{2} \dot{+} \mathbf{2}^5) + \lambda_0(\mathbf{2}) = 5.$$

From $\mathbf{L} = \mathbf{L}^\partial$, we get $\lambda_1(\mathbf{L}) = 5$. However, by Proposition 3.2, $\lambda(\mathbf{L}) \geq \lambda(\mathbf{2} \dot{+} \mathbf{2}^5)$. Since $\mathbf{2}^5$ has MPP with $\lambda(\mathbf{2}^5) = \lambda_1(\mathbf{2}^5)$, we get $\mathbf{2} \dot{+} \mathbf{2}^5$ has MPP. So, $\lambda(\mathbf{2} \dot{+} \mathbf{2}^5) = 6$. Consequently, $\mathbf{L}$ has does not have the NPP. □

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