International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 827–831

Zero-sum Ramsey number for non-cyclic group

Domingo Quiroz[1], **David Coronado**[2]

[1]Escuela Superior Politécnica del Litoral
FCNM, Campus Gustavo Galindo Km. 30.5 Vía Perimetral
P. O. Box 09-01-5863
Guayaquil, Ecuador

[2]Departamento de Computación y Tecnología de la Información
Universidad Simón Bolívar
Ap. 89000
Caracas 1080-A, Venezuela

email: daquiroz@espol.edu.ec

(Received March 26, 2021, Revised September 3, 2021 and November 23, 2021, Accepted December 2, 2021)

Abstract

Let G be a graph with n edges and let H be a finite abelian group such that the order of each element of H divides n. Let $R(G,H)$ denote the minimum integer t such that for every function $f : E(K_t) \to H$ there is a copy of G in K_t with the property that $\sum_{e\in E(G)} f(e) = 0$. We prove that for a positive integer r, if $H = \mathbb{Z}_n^r$ is the abelian group of all vectors of length r over $\mathbb{Z}_n$, then

$$R(G,2^r) \leq R(G,\mathbb{Z}_n^r), \tag{0.1}$$

where $R(G,2^r)$ is the smallest integer N such that for every 2^r-coloring of the edges of the complete graph K_N, there is a monochromatic copy of G. Moreover, we shall consider the bounds to $R(G,\mathbb{Z}_n^r)$ when G the star of n edges.

Key words and phrases: Ramsey numbers, zero sum, stars.
AMS (MOS) Subject Classifications: 05B10, 11B13.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

1 Introduction

Let G be a simple graph with finite vertex set $V(G)$ and edge set $E(G)$. Throughout this paper $n \geq k \geq r \geq 2$ denote positive integers and $n = |E(G)|$. For integers $a, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Recall that the Ramsey number $R(G, k)$ of a graph G is the smallest integer N such that for every k-coloring of the edges of the complete graph K_N, i.e., a function $c : E(K_N) \to [1, k]$, there is a subgraph G' in K_N isomorphic to G, where $c(E(G')) = \{i\}$ for an $i \in [1, k]$. The graph G' is called a monochromatic copy of G. Information on Ramsey numbers can be found in ongoing survey by Radziszowski [6].

Bialostocki and Dierker in [1] introduce a variant of the Ramsey number, concerned with coloring the edges of complete graphs with the elements of a cyclic group, and the appearance of zero-sum substructures (instead of monochromatic substructures). For a reference about most of the zero sum Ramsey results in the context of graph theory and comprehensive literature see the survey by Caro [3] and a recent result can be found in [5]. We are interested to know this variant when coloring a complete graph with the elements of an abelian group non-cyclic.

Definition 1.1. *Let H be a finite abelian group.* The zero-sum Ramsey number $R(G, H)$ *is the minimum integer t such that for every H-coloring $f : E(K_t) \to H$ of its edges, the complete graph K_t has a copy G' of G with the property that*

$$\sum_{e \in E(G')} f(e) = 0.$$

The graph G' is called a zero-sum copy of G.

The existence of the zero-sum Ramsey number for finite abelian groups is given by the following proposition.

Proposition 1.2. *Let G be a simple graph with $n > 0$ edges and let H be a finite abelian group. The zero-sum Ramsey number $R(G, H)$ exists if and only if the order of each element of H divides n.*

Proof: Let us suppose that there exist an $a \in H$ such that the order of a does not divide n. Then for any t the constant coloring $f : E(K_t) \to H$ given by $f(e) \equiv a$ avoids a zero-sum copy of G.

On the other hand, supposing the order of each element of H divides n, by Ramsey's theorem, for all t sufficiently large, for any coloring of $E(K_t)$

with $|H|$ or fewer colors there will be a monochromatic copy of G in K_t. Therefore, for t sufficiently large, for every H-coloring of the edges of Kt, some copy of G in Kt will be monochromatic. By hypothesis, the order of the constant color on the edges of that copy divides n, so the sum of the colors on the edges of the copy is zero. Thus $R(G, H)$ is well defined when the order of each element of H divides n. ■

The main concern of this paper establishes a relation between the zero-sum number Ramsey of a graph G coloring with $H = \mathbb{Z}_n^r$, the abelian group of all vectors of length r over $\mathbb{Z}_n$, and the monochromatic Ramsey number of a graph G coloring with 2^r colors. Moreover, we shall consider bounds to the zero sum Ramsey number of the star of n edges coloring with $\mathbb{Z}_n^r$.

Several preliminary definitions and results are needed in order to be able to deal with this problem. From combinatorial number theory, denote by the constant $s(\mathbb{Z}_n^r)$ to be smallest positive integer l such that every sequence S over $\mathbb{Z}_n^r$ of length $|S| \geq l$ has a zero sum subsequence T of length $|T| = n$. The existence of the best upper bound was proven in [4].

Theorem 1.3. *Let $H = \mathbb{Z}_n^r$ with $n \geq 2$ and $r \in \mathbb{N}$. Let $c \in \mathbb{N}$ such that for all prime p with $p|n$, we have $s(\mathbb{Z}_p^r) \leq c(p-1)+1$. Then*

$$s(H) \leq c(n-1)+1. \tag{1.2}$$

2 Main Results

Theorem 2.1. *Let $r \geq 2$ be a positive integer and let G be a graph with n edges. Then*

$$R(G, 2^r) \leq R(G, \mathbb{Z}_n^r). \tag{2.3}$$

Proof: We write N for the value of zero sum Ramsey number $R(G, \mathbb{Z}_n^r)$. We shall prove that for every coloration of the edges of K_N with 2^r colors, there is a monochromatic copy of G in K_N. First, let us recall that any integer $b \in [0, 2^r - 1]$ has a unique representation in the form

$$b = a_{r-1} \cdot 2^{r-1} + a_{r-2} \cdot 2^{r-2} + \cdots + a_1 \cdot 2 + a_0,$$

where $a_j \in \{0, 1\}$ for all $j \in [0, r-1]$. For $r \geq 2$, let $g : [0, 2^r - 1] \longrightarrow \mathbb{Z}_n^r$ be defined by $g(b) = (a_{r-1}, a_{r-2}, \ldots, a_1, a_0)$, where the a_i's are the coefficients of the binary representation of b. Now, we choose $f : E(K_N) \longrightarrow [0, 2^r - 1]$ to be a 2^r coloring of the edges of K_N, and we may induce a coloration $c : E(K_N) \longrightarrow \mathbb{Z}_n^r$, of the edges of K_N onto $\mathbb{Z}_n^r$, where c is given by $c = g \circ f$.

It follows from the definition of $R(G, \mathbb{Z}_n^r)$, that there is a copy G in K_N, with edge set $\{w_1, w_2, \ldots, w_n\}$, such that (with an abuse of notation)

$$\sum_{w \in E(G)} c(w) = c(w_1) + c(w_2) + \cdots + c(w_n) \equiv (0, 0, \ldots, 0) \mod n. \quad (2.4)$$

For $j = 1, 2, \ldots, n$, consider $b_j \in [0, 2^r - 1]$ such that $b_j = f(w_j)$ and set $(a_{r-1}^j, \ldots, a_1^j, a_0^j)$ to be the binary representation of b_j, writing

$$c(w_j) = (a_{r-1}^j, a_{r-2}^j. \ldots, a_1^j, a_0^j).$$

It follows, by (2.4), that for every $i \in [0, r-1]$, we have

$$\sum_{j=1}^{n} a_i^j \equiv 0 \mod n. \quad (2.5)$$

Let us define the set $B_s = \{t : a_s^t = 1\}$ for every $s \in [0, r-1]$. Since $a_s^t \in \{0, 1\}$ for any s and t, we must have, according to (2.5), that either

$$|B_s| = n \quad \text{or} \quad |B_s| = 0.$$

Therefore each b_j has the same binary representation. In consequence we must have $f(w_1) = \cdots = f(w_n)$, which is a monochromatic coloring of G. ■

Theorem 2.2. *Let n be an odd integer and $c, r \in \mathbb{N}$ such that for all prime p with $p|n$, we have $s(\mathbb{Z}_p^r) \leq c(p-1) + 1$. Then*

$$2^r(n-1) + 2 \leq R(K_{1,n}, \mathbb{Z}_n^r) \leq c(n-1) + 2. \quad (2.6)$$

Proof: For the lower bound consider Theorem 2.1 and the Ramsey number of the star $K_{1,n}$ with n odd, proved in [2]:

$$R(K_{1,n}, \mathbb{Z}_n^r) \geq R(K_{1,n}, 2^r) = 2^r(n-1) + 2.$$

For the upper bound we choose a coloring of the edges $K_{c(n-1)+2}$ into $\mathbb{Z}_n^r$. Since each vertex in $K_{c(n-1)+2}$ has degree $c(n-1)+1$, then by hypothesis and Theorem 1.3 there exists a subset of n edges incident with a vertex whose sum is 0 in $\mathbb{Z}_n^r$. ■

Acknowledgment

The authors are thankful to the referees for a number of comments that helped to improve the presentation of the manuscript.

References

[1] A. Bialostocki, P. Dierker, *On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings*, Discrete Math., **110,** nos. 1-3, (1992), 1–8.

[2] S. A. Burr, J. A. Roberts, *On Ramsey number of stars*, Util. Math., **4,** (1973), 217–220.

[3] Y. Caro, *Zero-sum problems– A survey,* Discrete Math., **152,** (1996), 93–113.

[4] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, L. Rackham, *Zero-sum problems in finite abelian groups and affine caps*, Quart. J. Math., **58,** (2007), 159–186

[5] D. Coronado, D Quiroz, *The zero sum Ramsey numbers of the join of two graphs*, Inter. J. Math. Computer Sci., **15,** no. 2, (2020) 627–632.

[6] S. Radziszowski. *Small Ramsey numbers*, Electronic Journal of Combinatorics, DS1: March 3, 2017.