International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 865–874

On the Eigensharp of Corona Product

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(Received November 2, 2021, Accepted December 18, 2021)

Abstract

In this paper, we study the eigensharp property for the corona product of a graph G with a complete graph K_m, the corona product of a complete graph with a regular graph G, the corona product of a graph G with a complete bipartite graph $K_{n,n}$ and the corona product of a complete bipartite graph with a regular graph G.

1 Introduction

Throughout this paper, all graphs are finite undirected simple graphs. For a graph $G = (V(G), E(G))$, $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. It is assumed that a graph G has a non-empty vertex set. The order of a graph G is equal to the cardinality of $V(G)$ and is denoted by $|G|$.

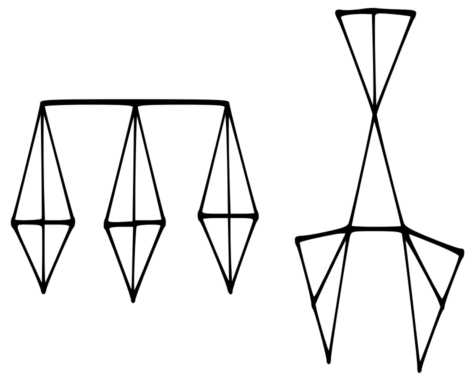
Key words and phrases: Complete Graph, Complete bipartite graph, Biclique, Biclique partition number, Corona product, Eigensharp graph.
AMS (MOS) Subject Classifications: 05C50, 05C70, 05C76.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

If $\lambda_i, 1 \leq i \leq k$ are the distinct eigenvalues of the adjacency matrix $A(G)$ with multiplicity m_i, then $\sigma(G) = ((\lambda_1)^{m_1}, (\lambda_2)^{m_2}, ..., (\lambda_k)^{m_k})$ is called the spectrum of G. In $\sigma(G)$, the number of positive and negative eigenvalues is denoted by $a_+(G)$ and $a_-(G)$, respectively. It can be proved that $a_+(G) > 0$ and $a_-(G) > 0$, for any non null graph G. The largest eigenvalue of $A(G)$ is called the spectral radius of a graph G and is denoted by $\rho(G)$. A clique is a complete subgraph, and a biclique is a complete bipartite subgraph.

A biclique partition of a graph G is the partition of the set of edges $E(G)$ into k disjoint biclique subgraphs. The biclique partition number is the minimal cardinality of a biclique partition of G, denoted by $bp(G)$. In 1972, Graham and Pollak [8] introduced the biclique partition number. Several studies on the subject have been published since then (See [2], [3] and [4]). Witsenhausen proved that for a graph G the biclique partition number $bp(G) \geq \max\{a_+(G),\ a_-(G)\}$ (see [9]).

A graph G is called eigensharp if $bp(G) = \max\{a_+(G), a_-(G)\}$ while a graph G is called an almost eigensharp when $bp(G) = \max\{a_+(G),\ a_-(G)\}$ $+1$, see [3]. There are certain families of eigensharp graphs. Complete graphs K_n are eigensharp, see [7]. Complete bipartite graphs $K_{n,m}$ are also an eigensharp graph, where $\sigma(K_{n,m}) = ((0)^{m-2}, (\sqrt[2]{mn})^1, (-\sqrt[2]{mn})^1)$ and $bp(K_{n,m}) = 1$. In particular, the star graphs $S_n = K_{1,n-1}$ is a tree graph with n vertices, one of them is called the generator of S_n, is an eigensharp graph. Also, cycle graphs C_n with $n \neq 4k$ and $k > 2$ are eigensharp graphs [9].

Graph products are useful in constructing many important classes of graphs. The corona product of two graphs was first introduced by Frucht and Harary [6]. For any two graphs G_1 and G_2 with orders n and m respectively, the corona product $G_1 \odot G_2$ is defined as the graph consisting of G_1 and n copies of G_2, where the i-th vertex of G_1 is adjacent to each vertex in the i-th copy of G_2 for each $i = 1, 2, ..., n$. It is clear that $G_1 \odot G_2$ and $G_2 \odot G_1$ are not the same graph. The following Figure shows $P_3 \odot K_3$ and $K_3 \odot P_3$.



$P_3 \odot K_3$ $K_3 \odot P_3$

In this paper, we discuss the eigensharp property for several classes of corona product of two graph. In section 2, we study the eigensharp property of the corona product of a graph G of order n with a complete graph K_m, and the corona product of K_m with a regular graph G of order n. In section 3, we deal with the eigensharp property of the corona product of a graph G of order m with a complete bipartite graph $K_{n,n}$ and the corona product of complete bipartite graph $K_{n,n}$ with a regular graph G of order m.

The following theorem specifies the spectrum of the corona product $G_1 \odot G_2$ when G_2 is a regular graph.

Theorem 1.1. *[8] Let G_1 be a graph of order n. Let $\lambda_i, 1 \leq i \leq n$ be the eigenvalues of $A(G_1)$ and let G_2 be an r-regular graph of order m with $\mu_1 \leq \mu_2 \leq ... \leq \mu_m = r$ denoting the eigenvalues of $A(G_2)$. Then*

$$\sigma(G_1 \odot G_2) = \left((\mu_1)^n, (\mu_2)^n, ..., (\mu_{m-1})^n, \left(\frac{1}{2} \left(\lambda_i + r \pm \sqrt{(r-\lambda_i)^2 + 4m} \right) \right)^1 \right).$$

2 The Corona Product Graphs with K_m

In this section, we study the eigensharp property of the corona product of a graph G of order n with a complete graph K_m and the corona product of K_m with a regular graph G of order n.

In the following Theorem, we show that $G \odot K_1$ is eigensharp for any graph G.

Theorem 2.1. *The graph $G \odot K_1$ is eigensharp.*

Proof. Let $V(G) = \{u_1, ...u_n\}$ be the set of vertices of G and let v be the vertex of K_1. Assume that $\lambda_i, 1 \le i \le n$ are the eigenvalues of $A(G)$. Then, by Theorem 1.1,

$$\sigma(G \odot K_1) = \left(\left(\frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2} \right)^1 \right).$$

Hence, $a_+ (G \odot K_1) = a_- (G \odot K_1) = n$. So $bp\,(G \odot K_1) \ge n$. On the other hand, let $W = \{S_1, ..., S_n\}$ be the set of n disjoint stars in K_n generated by $u_1, ..., u_n$ respectively. Then $S_i \ne \phi$, for all $1 \le i \le n$ since it contains at least the edge $u_i v$. Thus W forms a biclique partition of $G \odot K_1$ with minimum cardinality equals n. Hence $bp\,(G \odot K_1) = n$ and so $G \odot K_1$ is an eigensharp graph. □

For a regular graph G, we now show that $K_1 \odot G$ is an eigensharp graph whenever G is eigensharp graph with $bp(G) = a_- (G)$.

Theorem 2.2. *If G is r-regular eigensharp graph with $bp(G) = a_-(G)$, then $K_1 \odot G$ is an eigensharp graph.*

Proof. Let $V(G) = \{u_1, ...u_n\}$ be the set of vertices of G and let v be the vertex of K_1. The graph $K_1 \odot G$ is given by

$$\begin{aligned} V\,(K_1 \odot G) &= \{v, u_1, ...u_n\} \\ E\,(K_1 \odot G) &= E\,(G) \cup \{vu_i : 1 \le i \le n\}. \end{aligned}$$

Then, $a_- (K_1 \odot G) = a_-(G) + 1$ and $a_+ (K_1 \odot G) = a_+(G) - 1$ (see [3]). Since G is an eigensharp graph with $bp(G) = a_- (G)$, we have $a_- (K_1 \odot G) = \max\{a_- (K_1 \odot G), a_+ (K_1 \odot G)\}$. Hence, $bp(K_1 \odot G) \ge bp(G) + 1$. On the other hand, we can partition the edges of $K_1 \odot G$ by $bp(G) + 1$ bicliques graphs; the set of complete bipartite subgraphs partitions G with the star generated by the vertex v. Therefore, $K_1 \odot G$ is an eigensharp graph. □

In general, if G is a regular graph with $bp(G) \ne a_-(G)$, then $K_1 \odot G$ might not be an eigensharp graph. For example, consider the graph $K_1 \odot C_8$. Then $bp(K_1 \odot C_8) = 6$ while by simple calculation we can show that $\max\{a_-(K_1 \odot C_8), a_+(K_1 \odot C_8)\} = a_+(K_1 \odot C_8) = 4$.

In the following theorem, we now compute the biclique partition number of $G \odot K_m$.

Theorem 2.3. *For a graph G, $bp(G \odot K_m) = m\,|G|$.*

Proof. Let G be a graph of order n. The graph $G \odot K_m$ is formed by a copy of G and n copies of K_m. Let $V(G) = \{u_1, u_2, ..., u_n\}$ be the vertex set of G. For $1 \le i \le n$, let $W = \{v_{1,i}, v_{2,i}, ..., v_{m,i}\}$ be the set of vertices of the i-th copy of K_m. Then, the graph $G \odot K_m$ is given by

$$V(G \odot K_m) = \{u_k, v_{j,i} : 1 \le k, i \le n, 1 \le j \le m\},$$

where u_i is adjacent to each vertex in the i-th copy of K_m, and v_{j,i_r}, v_{j,i_s} are non adjacent for all $i_r \neq i_s$. In $G \odot K_m$, we have n disjoint cliques, denoted by H_i, each of them has an order $m+1$ and is induced by $V(H_i) = \{u_i, v_{1,i}, v_{2,i}, ..., v_{m,i}\}$. Thus, by Graham and Pollak theorem, $bp(H_i) = m$. Now, let

$$x \in \{u_i, v_{1,i}, v_{2,i}, ..., v_{m-1,i}\}$$

and let S_x be the star that generated by x. Then, without loss of generality, $M_i = \{S_{u_i}, S_{v_{1,i}}, S_{v_{2,i}}, ..., S_{v_{m-1,i}}\}$ is a biclique partition of the set of edges of the clique H_i with cardinality m. Since all edges of G are covered, $bp(G \odot K_m) = \sum_{i=1}^{n} |M_i| = m\,|G|$. □

In the following theorem, we show that there is a relation between $\rho(G)$ and being the graph $G \odot K_m$ is an eigensharp.

Theorem 2.4. *The graph $G \odot K_m$ is eigensharp if and only if $\rho(G) < \frac{m}{m-1}$ for each $m > 1$.*

Proof. Let G be a graph of order n and let $\lambda_i, 1 \le i \le n$ be the eigenvalues of $A(G)$. Then, by Theorem 1.1, we have

$$\sigma(G \odot K_m) = \left((-1)^{n(m-1)}, \left(\tfrac{1}{2} \left(\lambda_i + m - 1 \pm \sqrt{(m-1-\lambda_i)^2 + 4m} \right) \right)^1 \right).$$

By Theorem 2.3, $bp(G \odot K_m) = mn$. Assume that $G \odot K_m$ is an eigensharp graph. Then $mn = \max\{a_+(G \odot K_m), a_-(G \odot K_m)\}$. If $m = 2$ or $m = 3$ and $bp(G \odot K_m) = a_+(G \odot K_m) = nm$, then $\lambda_i + m - 1 - \sqrt{(m-1-\lambda_i)^2 + 4m} > 0$, for all $1 \le i \le n$, which implies that $\lambda_i > 1$. This means that $a_-(G) = 0$ which is a contradiction. Thus $bp(G \odot K_m) = a_-(G \odot K_m)$, for $m = 2$ or $m = 3$. Moreover, if $m > 3$, then $a_-(G \odot K_m) > a_+(G \odot K_m)$. Therefore, $mn = a_-(G \odot K_m)$, for each $m > 1$. For each i, $\lambda_i + m - 1 + \sqrt{(m-1-\lambda_i)^2 + 4m} > 0$, we have n positive eigenvalues. Then $\lambda_i + m - 1 - \sqrt{(m-1-\lambda_i)^2 + 4m}$

has to be a negative number for all $1 \leq i \leq n$. If $\lambda_i + m - 1 < 0$ for all $1 \leq i \leq n$, then it is clear that $\rho(G) < \frac{m}{m-1}$.
If $\lambda_i + m - 1 > 0$ for some $1 \leq i \leq n$, then $\lambda_i + m - 1 - \sqrt{(m-1-\lambda_i)^2 + 4m} < 0$ if and only if $\lambda_i < \frac{m}{m-1}$ for all $i : 1 \leq i \leq n$. Thus $\rho(G) < \frac{m}{m-1}$.

Now if $\rho(G) < \frac{m}{m-1}$, then $\lambda_i + m - 1 - \sqrt{(m-1-\lambda_i)^2 + 4m} < 0$ for all $1 \leq i \leq n$, so $a_-(G \odot K_m) = mn > a_+(G \odot K_m) = n$. Thus $bp(G \odot K_m) = mn = a_-(G \odot K_m) = \max\{a_+(G \odot K_m), a_-(G \odot K_m)\}$. Hence, $G \odot K_m$ is an eigensharp graph. □

Next, we discuss the graph $K_m \odot G$. We specify a condition on G that leads $K_m \odot G$ to be almost eigensharp.

To begin with, we present the following theorem for the biclique partition number of $K_m \odot G$ in general.

Theorem 2.5. *For the graph $K_m \odot G$, $bp(K_m \odot G) = m(bp(G) + 1)$ for $m \geq 1$.*

Proof. Let G be a graph of order n. Then, $K_m \odot G$ consists of K_m and m copies of G; namely, $G_i : 1 \leq i \leq m$. Now, in order to define the biclique partition number of $K_m \odot G$, the values $bp(G)$ and $bp(K_m)$ must be taken into account. Let $V(K_m) = \{u_1, u_2, ..., u_m\}$ be the set of vertex of K_m. Then, $bp(K_m) = m - 1$. Let S_i be the star in K_m generated by u_i. Then, without loss of generality, we have $E(K_m) = \cup_{i=1}^{m-1} S_i$. Also, since $\{G_i\}_{i=1}^{n}$ is a family of disjoint graphs with $bp(G_i) = bp(G)$ for each i, we have $bp(K_m \odot G) \geq (m-1) + mbp(G)$. Now, since exceeding u_m will result in the edges connecting u_m and G_m are not being covered, S_m; the star in K_m generated by u_m, must be included in the partition. Hence $bp(K_m \odot G) = [(m-1) + mbp(G)] + 1 = m(bp(G) + 1)$. □

Theorem 2.6. *If G is an r-regular eigensharp graph with $bp(G) = a_-(G)$, then $K_m \odot G$ is an eigensharp graph when $r = 1$; otherwise, it is an almost eigensharp graph for each $m > 1$.*

Proof. Let G be a graph of order n. Assume that $\lambda_i, 1 \leq i \leq n$ are the eigenvalues of $A(G)$ with $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = r$. Then, by Theorem 1.1

$$\sigma(K_m \odot G) = \left((\lambda_1)^m, (\lambda_2)^m, ..., (\lambda_{n-1})^m, \left(\frac{1}{2}\left(-1 + r \pm \sqrt{(r+1)^2 + 4m}\right)\right)^{m-1}, \left(\frac{1}{2}\left((m-1) + r \pm \sqrt{(r-(m-1))^2 + 4m}\right)\right)^{1} \right).$$

Suppose that $t = a_+(G) < k = a_-(G)$ and $bp(G) = k$. By Theorem 2.5, $bp(K_m \odot G) = m(k+1)$.

If $r = 1$, then $(m-1) + r + \sqrt{(r-(m-1))^2 + 4m} > 0$ and $-1 + r + \sqrt{(r+1)^2 + 4m} > 0$. Also, $(m-1) + r - \sqrt{(r-(m-1))^2 + 4m} < 0$ and $-1 + r - \sqrt{(r+1)^2 + 4m} < 0$. So $a_-(K_m \odot G) = m + mk$ and $a_+(K_m \odot G) = m(t-1)+m$. Then, $bp(K_m \odot G) = m(k+1) = \max\{a_+(K_m \odot G), a_-(K_m \odot G)\}$. Therefore, $K_m \odot G$ is an eigensharp graph.

If $r > 1$, then we have $(m-1) + r + \sqrt{(r-(m-1))^2 + 4m} > 0$, $(m-1) + r - \sqrt{(r-(m-1))^2 + 4m} > 0$ and $-1 + r + \sqrt{(r+1)^2 + 4m} > 0$. Also, $-1 + r - \sqrt{(r+1)^2 + 4m} < 0$. So $a_-(K_m \odot G) = m - 1 + mk$ and $a_+(K_m \odot G) = m(t-1)+m+1$. Then, $bp(K_m \odot G) = m(k+1) = \max\{a_+(K_m \odot G), a_-(K_m \odot G)\} + 1$. Hence $K_m \odot G$ is an almost eigensharp graph. □

3 The Corona Product Graphs With $K_{n,n}$

In this section, we study the eigensharp property of the corona product of a graph G of order m with a complete bipartite graph $K_{n,n}$ and the corona product of a complete bipartite graph $K_{n,n}$ with a regular graph G of order m.

Theorem 3.1. *Let G be a graph of order m. Then $bp(G \odot K_{n,n}) = 2\,|G|$.*

Proof. Let G be a graph of order m and let $V(G) = \{u_1, u_2, ..., u_m\}$ be the vertex set of G. Then, $G \odot K_{n,n}$ is formed by a copy of G and m copies of $K_{n,n}$; namely, $K^i_{n,n} : 1 \le i \le m$, such that u_i in G is adjacent to each vertex in $K^i_{n,n}$.

In $G \odot K_{n,n}$, $\{K^i_{n,n}\}_{i=1}^m$ is a family of disjoint bicliques. So $\{u_i \odot K_{n,n}\}_{i=1}^m$ is a family of disjoint subgraphs with $bp(u_i \odot K_{n,n}) = 2$, for each $1 \le i \le m$. In fact, the biclique partition covering of $u_i \odot K^i_{n,n}$ with minimum cardinality is the star S_{u_i} generated by u_i with a biclique $K^i_{n,n}$ for $1 \le i \le m$. This implies that the biclique partition of minimum cardinality of $G \odot K_{n,n}$ is a union of all families of stars $\{S_{u_i}\}_{i=1}^m$ with the family $\left\{K^i_{n,n}\right\}_{i=1}^m$. Therefore, we have $bp\,(G \odot K_{n,n}) = 2m = 2\,|G|$. □

Theorem 3.2. *Let G be a graph of order m. Then the graph $G \odot K_{n,n}$ is eigensharp if and only if $\rho(G) < 2$ for each $n > 1$.*

Proof. Let G be a graph of order m and let $\lambda_i, 1 \leq i \leq m$ be the eigenvalues of $A(G)$. Then, by Theorem 1.1, we have
$\sigma(G \odot K_{n,n}) = \left((-n)^m, (0)^{2m(n-1)}, \left(\frac{1}{2}\left(\lambda_i + n \pm \sqrt{(n-\lambda_i)^2 + 8n}\right)\right)^1\right).$
By Theorem 3.1, $bp(G \odot K_{n,n}) = 2|G|$. Assume that $G \odot K_{n,n}$ is an eigensharp graph. Then $2|G| = \max\{a_+(G \odot K_{n,n}), a_-(G \odot K_{n,n})\}$. If $bp(G \odot K_{n,n}) = a_+(G \odot K_{n,n}) = 2|G|$, then $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n} > 0$ for all $1 \leq i \leq m$ which implies that $\lambda_i > 2$. This means that $a_-(G) = 0$ which is a contradiction. Thus $bp(G \odot K_{n,n}) = 2|G| = a_-(G \odot K_m)$, so $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n}$ has to be negative number for all $1 \leq i \leq m$. Now, if $\lambda_i + n < 0$ then it is clear that $\lambda_i < 2$ for all $1 \leq i \leq m$. If $\lambda_i + n > 0$ for some $1 \leq i \leq m$, then the inequality $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n} < 0$ is true if and only if $\lambda_i < 2$ for all $1 \leq i \leq m$, so $\rho(G) < 2$. Now, if $\rho(G) < 2$, then $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n} < 0$ for all $1 \leq i \leq m$. Thus, $bp(G \odot K_{n,n}) = 2m = a_-(G \odot K_{n,n}) = \max\{a_+(G \odot K_{n,n}), a_-(G \odot K_{n,n})\}$ which implies that $G \odot K_{n,n}$ is an eigensharp graph. □

Theorem 3.3. *For the graph $K_{n,n} \odot G$, $bp(K_{n,n} \odot G) = 2n(bp(G) + 1)$.*

Proof. Let $V(K_{n,n}) = \{u_1, u_2, ..., u_{2n}\}$ be the vertex set of $K_{n,n}$. Then, $K_{n,n} \odot G$ is formed by a copy of $K_{n,n}$ and $2n$ copies of $G^i : 1 \leq i \leq 2n$, such that each vertex u_i in $K_{n,n}$ is adjacent to each vertex in the $i-th$ copy $G_i : 1 \leq i \leq 2n$. So, we have a family of disjoint subgraph; namely, $\{u_i \odot G_i\}_{i=1}^{2n}$. Thus, $bp(u_i \odot G_i) = 1 + bp(G_i)$ where the biclique partition covering of $u_i \odot G_i$, with minimum cardinality being a star generated by $u_i : 1 \leq i \leq 2n$ and a biclique partition of minimum cardinality of G_i (see [3]). This shows that the biclique partition of minimum cardinality of $K_{n,n} \odot G$ is a union of the family of stars generated by $\{u_i\}_{i=}^{2n}$ with the biclique partition of minimum cardinality of $G_i : 1 \leq i \leq 2n$. Therefore, we have $bp\left(G \odot K_{n,n}\right) = 2n + 2nbp(G)$. □

Theorem 3.4. *Suppose G is an r-regular eigensharp graph of order m with $bp(G) = a_-(G)$.*
(i) If $n < \frac{m}{r}$, then $K_{n,n} \odot G$ is an eigensharp graph.
(ii) If $n \geq \frac{m}{r}$, then $K_{n,n} \odot G$ is almost eigensharp graph.

Proof. Let G be a graph of order m. Assume that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m = r$ are the eigenvalues of $A(G)$. Then, by Theorem 1.1,

$$\sigma(K_{n,n}\odot G)=\left(\left(\tfrac{1}{2}\left(n+r\pm\sqrt{(r-n)^2+4m}\right)\right)^1,\left(\tfrac{1}{2}\left(r\pm\sqrt{r^2+4m}\right)\right)^{2n-2},\right.$$
$$\left.\left(\tfrac{1}{2}\left(-n+r\pm\sqrt{(r+n)^2+4m}\right)\right)^1,\lambda_1^{2n},\lambda_2^{2n},...,\lambda_{m-1}^{2n}\right).$$

Assume that $t=a_+(G)<k=a_-(G)$ and $bp(G)=k$. By Theorem 3.3, $bp(K_{n,n}\odot G)=2n(k+1)$.

i) If $n<\frac{m}{r}$, then $-n+r-\sqrt{(r+n)^2+4m}<0$, $n+r-\sqrt{(r-n)^2+4m}<0$ and $r-\sqrt{r^2+4m}<0$. Also, $-n+r+\sqrt{(r+n)^2+4m}>0, n+r+\sqrt{(r-n)^2+4m}>0$ and $r+\sqrt{r^2+4m}>0$. So, $a_-(K_{n,n}\odot G)=2nk+2n$ and $a_+(K_{n,n}\odot G)=2n+2n(t-1)=2nt$. Therefore, $bp(K_{n,n}\odot G)=\max\{a_-(K_{n,n}\odot G),a_+(K_{n,n}\odot G)\}=2n(k+1)$. Hence the graph $K_{n,n}\odot G$ is eigensharp.

ii) If $n\geq\frac{m}{r}$, then $-n+r-\sqrt{(r+n)^2+4m}<0$ and $r-\sqrt{r^2+4m}<0$. Also, $n+r-\sqrt{(r-n)^2+4m}\geq 0$, $-n+r+\sqrt{(r+n)^2+4m}>0$, $n+r+\sqrt{(r-n)^2+4m}>0$ and $r+\sqrt{r^2+4m}>0$. So $a_-(K_{n,n}\odot G)=2nk+2n-1$ and $a_+(K_{n,n}\odot G)=2n+2n(t-1)+1=2nt+1$. Thus, $\max\{a_-(K_{n,n}\odot G),a_+(K_{n,n}\odot G)\}=2n(k+1)-1$. Therefor, $bp(K_{n,n}\odot G)=\max\{a_-(K_{n,n}\odot G),a_+(K_{n,n}\odot G)\}+1$ and hence the graph $K_{n,n}\odot G$ is almost eigensharp. □

Acknowledement. The third author is supported by the Scientific Research and Graduate Studies at Yarmouk University.

References

[1] A. E. Brouwer, W. H. Haemers, Spectra of graphs, Springer, New York, 2012.

[2] D. Gregory, B. Shader, V. Watts, Biclique decomposition and Hermitian rank, Linear Algebra and its Application, (1999), 267–280.

[3] E. Ghorbani, H. R. Maimani, On eigensharp and almost eigensharp graphs, Linear Algebra and its Applications, (2008), 2746–2753.

[4] E. Rawshdeh, H. AL-Ezeh, The biclique partition number of some important graph, Italian Journal of Pure and Applied Mathematics, **41,** (2019), 274–283.

[5] E. V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, Linear Algebra and its Application, (1968), 73–81.

[6] Frank Harary, Graph theory, Addison-Wesley , Reading, MA, 1970.

[7] R. L. Graham, H. O. Pollak, On the addressing problem for loop switching, Bell System Tech. J., (1971), 2495–2519.

[8] S. Barik, S. Pati, B. K. Sarmat, The Spectrum of the Corona of Two Graphs, Society for Industrial and Applied Mathematics, (2007), 47–56.

[9] T. Kratzke, B. Reznick, D. West, Eigensharp graphs: decomposition into complete bipartite subgraphs, Trans. Amer. Math. Soc., (1988), 637–653.