On the Eigensharp of Corona Product

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(Received November 2, 2021, Accepted December 18, 2021)

Abstract

In this paper, we study the eigensharp property for the corona product of a graph \( G \) with a complete graph \( K_m \), the corona product of a complete graph with a regular graph \( G \), the corona product of a graph \( G \) with a complete bipartite graph \( K_{n,n} \) and the corona product of a complete bipartite graph with a regular graph \( G \).

1 Introduction

Throughout this paper, all graphs are finite undirected simple graphs. For a graph \( G = (V(G), E(G)) \), \( V(G) \) is the set of vertices and \( E(G) \) is the set of edges of \( G \). It is assumed that a graph \( G \) has a non-empty vertex set. The order of a graph \( G \) is equal to the cardinality of \( V(G) \) and is denoted by \( |G| \).

Key words and phrases: Complete Graph, Complete bipartite graph, Biclique, Biclique partition number, Corona product, Eigensharp graph.

AMS (MOS) Subject Classifications: 05C50, 05C70, 05C76.

ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net
If \( \lambda_i, 1 \leq i \leq k \) are the distinct eigenvalues of the adjacency matrix \( A(G) \) with multiplicity \( m_i \), then \( \sigma(G) = ((\lambda_1)^{m_1}, (\lambda_2)^{m_2}, ..., (\lambda_k)^{m_k}) \) is called the spectrum of \( G \). In \( \sigma(G) \), the number of positive and negative eigenvalues is denoted by \( a_+(G) \) and \( a_-(G) \), respectively. It can be proved that \( a_+(G) > 0 \) and \( a_-(G) > 0 \), for any non null graph \( G \). The largest eigenvalue of \( A(G) \) is called the spectral radius of a graph \( G \) and is denoted by \( \rho(G) \).

A biclique partition of a graph \( G \) is the partition of the set of edges \( E(G) \) into \( k \) disjoint biclique subgraphs. The biclique partition number is the minimal cardinality of a biclique partition of \( G \), denoted by \( bp(G) \). In 1972, Graham and Pollak [8] introduced the biclique partition number. Several studies on the subject have been published since then (See [2], [3] and [4]). Witsenhausen proved that for a graph \( G \) the biclique partition number \( bp(G) \geq \max\{a_+(G), a_-(G)\} \) (see [9]).

A graph \( G \) is called eigensharp if \( bp(G) = \max\{a_+(G), a_-(G)\} \) while a graph \( G \) is called an almost eigensharp when \( bp(G) = \max\{a_+(G), a_-(G)\} + 1 \), see [3]. There are certain families of eigensharp graphs. Complete graphs \( K_n \) are eigensharp, see [7]. Complete bipartite graphs \( K_{n,m} \) are also an eigensharp graph, where \( \sigma(K_{n,m}) = ((0)^{m-2}, (\sqrt{mn})^1, (-\sqrt{mn})^1) \) and \( bp(K_{n,m}) = 1 \). In particular, the star graphs \( S_n = K_{1,n-1} \) is a tree graph with \( n \) vertices, one of them is called the generator of \( S_n \), is an eigensharp graph. Also, cycle graphs \( C_n \) with \( n \neq 4k \) and \( k > 2 \) are eigensharp graphs [9].

Graph products are useful in constructing many important classes of graphs. The corona product of two graphs was first introduced by Frucht and Harary [6]. For any two graphs \( G_1 \) and \( G_2 \) with orders \( n \) and \( m \) respectively, the corona product \( G_1 \odot G_2 \) is defined as the graph consisting of \( G_1 \) and \( n \) copies of \( G_2 \), where the \( i \)-th vertex of \( G_1 \) is adjacent to each vertex in the \( i \)-th copy of \( G_2 \) for each \( i = 1, 2, ..., n \). It is clear that \( G_1 \odot G_2 \) and \( G_2 \odot G_1 \) are not the same graph. The following Figure shows \( P_3 \odot K_3 \) and \( K_3 \odot P_3 \).
In this paper, we discuss the eigensharp property for several classes of corona product of two graph. In section 2, we study the eigensharp property of the corona product of a graph $G$ of order $n$ with a complete graph $K_m$, and the corona product of $K_m$ with a regular graph $G$ of order $n$. In section 3, we deal with the eigensharp property of the corona product of a graph $G$ of order $m$ with a complete bipartite graph $K_{n,n}$ and the corona product of complete bipartite graph $K_{n,n}$ with a regular graph $G$ of order $m$.

The following theorem specifies the spectrum of the corona product $G_1 \circ G_2$ when $G_2$ is a regular graph.

**Theorem 1.1.** [8] Let $G_1$ be a graph of order $n$. Let $\lambda_i, 1 \leq i \leq n$ be the eigenvalues of $A(G_1)$ and let $G_2$ be an $r$-regular graph of order $m$ with $\mu_1 \leq \mu_2 \leq ... \leq \mu_m = r$ denoting the eigenvalues of $A(G_2)$. Then

$$\sigma(G_1 \circ G_2) = \left( (\mu_1)^n, (\mu_2)^n, ..., (\mu_{m-1})^n, \left( \frac{1}{2} (\lambda_i + r \pm \sqrt{(r - \lambda_i)^2 + 4m})^n \right) \right).$$

**2 The Corona Product Graphs with $K_m$**

In this section, we study the eigensharp property of the corona product of a graph $G$ of order $n$ with a complete graph $K_m$ and the corona product of $K_m$ with a regular graph $G$ of order $n$.

In the following Theorem, we show that $G \circ K_1$ is eigensharp for any graph $G$.

**Theorem 2.1.** The graph $G \circ K_1$ is eigensharp.
Proof. Let $V(G) = \{u_1, \ldots, u_n\}$ be the set of vertices of $G$ and let $v$ be the vertex of $K_1$. Assume that $\lambda_i, 1 \leq i \leq n$ are the eigenvalues of $A(G)$. Then, by Theorem 1.1,

$$\sigma(G \odot K_1) = \left(\frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2}\right)^1.$$ 

Hence, $a_+ (G \odot K_1) = a_- (G \odot K_1) = n$. So $bp (G \odot K_1) \geq n$. On the other hand, let $W = \{S_1, \ldots, S_n\}$ be the set of $n$ disjoint stars in $K_n$ generated by $u_1, \ldots, u_n$ respectively. Then $S_i \neq \emptyset$, for all $1 \leq i \leq n$ since it contains at least the edge $u_iv$. Thus $W$ forms a biclique partition of $G \odot K_1$ with minimum cardinality equals $n$. Hence $bp (G \odot K_1) = n$ and so $G \odot K_1$ is an eigensharp graph.

For a regular graph $G$, we now show that $K_1 \odot G$ is an eigensharp graph whenever $G$ is eigensharp graph with $bp(G) = a_- (G)$.

**Theorem 2.2.** If $G$ is $r$-regular eigensharp graph with $bp(G) = a_- (G)$, then $K_1 \odot G$ is an eigensharp graph.

*Proof.* Let $V(G) = \{u_1, \ldots, u_n\}$ be the set of vertices of $G$ and let $v$ be the vertex of $K_1$. The graph $K_1 \odot G$ is given by

$$V(K_1 \odot G) = \{v, u_1, \ldots, u_n\}$$
$$E(K_1 \odot G) = E(G) \cup \{vu_i: 1 \leq i \leq n\}.$$ 

Then, $a_- (K_1 \odot G) = a_- (G) + 1$ and $a_+ (K_1 \odot G) = a_+ (G) - 1$ (see [3]). Since $G$ is an eigensharp graph with $bp(G) = a_- (G)$, we have $a_- (K_1 \odot G) = \max\{a_- (K_1 \odot G) \cup a_+ (K_1 \odot G)\}$. Hence, $bp(K_1 \odot G) \geq bp(G) + 1$. On the other hand, we can partition the edges of $K_1 \odot G$ by $bp(G) + 1$ bicliques graphs; the set of complete bipartite subgraphs partitions $G$ with the star generated by the vertex $v$. Therefore, $K_1 \odot G$ is an eigensharp graph.

In general, if $G$ is a regular graph with $bp(G) \neq a_- (G)$, then $K_1 \odot G$ might not be an eigensharp graph. For example, consider the graph $K_1 \odot C_8$. Then $bp(K_1 \odot C_8) = 6$ while by simple calculation we can show that $\max\{a_- (K_1 \odot C_8), a_+ (K_1 \odot C_8)\} = a_+ (K_1 \odot C_8) = 4$.

In the following theorem, we now compute the biclique partition number of $G \odot K_m$. 
Theorem 2.3. For a graph $G$, $bp(G) = m |G|$.

Proof. Let $G$ be a graph of order $n$. The graph $G \odot K_m$ is formed by a copy of $G$ and $n$ copies of $K_m$. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of $G$. For $1 \leq i \leq n$, let $W = \{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\}$ be the set of vertices of the $i$-th copy of $K_m$. Then, the graph $G \odot K_m$ is given by

$$V(G \odot K_m) = \{u_k, v_{j,i} : 1 \leq k, i \leq n, 1 \leq j \leq m\},$$

where $u_i$ is adjacent to each vertex in the $i$-th copy of $K_m$, and $v_{j,i}$ is non adjacent for all $i_r \neq i_s$. In $G \odot K_m$, we have $n$ disjoint cliques, denoted by $H_i$, each of them has an order $m + 1$ and is induced by $V(H_i) = \{u_i, v_{1,i}, v_{2,i}, \ldots, v_{m,i}\}$. Thus, by Graham and Pollak theorem, $bp(H_i) = m$. Now, let

$$x \in \{u_i, v_{1,i}, v_{2,i}, \ldots, v_{m-1,i}\}$$

and let $S_x$ be the star that generated by $x$. Then, without loss of generality, $M_i = \{S_{u_1}, S_{v_{1,i}}, S_{v_{2,i}}, \ldots, S_{v_{m-1,i}}\}$ is a biclique partition of the set of edges of the clique $H_i$ with cardinality $m$. Since all edges of $G$ are covered, $bp(G \odot K_m) = \sum_{i=1}^{n} |M_i| = m |G|$.

In the following theorem, we show that there is a relation between $\rho(G)$ and being the graph $G \odot K_m$ is an eigensharp.

Theorem 2.4. The graph $G \odot K_m$ is eigensharp if and only if $\rho(G) < \frac{m}{m-1}$ for each $m > 1$.

Proof. Let $G$ be a graph of order $n$ and let $\lambda_i, 1 \leq i \leq n$ be the eigenvalues of $A(G)$. Then, by Theorem 1.1, we have

$$\sigma(G \odot K_m) = \left((-1)^{n(m-1)}, \left(\frac{1}{2} \left(\lambda_i + m - 1 \pm \sqrt{(m - 1 - \lambda_i)^2 + 4m}\right)\right)\right).$$

By Theorem 2.3, $bp(G \odot K_m) = mn$. Assume that $G \odot K_m$ is an eigensharp graph. Then $mn = \max\{a_+(G \odot K_m), a_-(G \odot K_m)\}$. If $m = 2$ or $m = 3$ and $bp(G \odot K_m) = a_+(G \odot K_m) = mn$, then $\lambda_i + m - 1 - \sqrt{(m - 1 - \lambda_i)^2 + 4m} > 0$, for all $1 \leq i \leq n$, which implies that $\lambda_i > 1$. This means that $a_-(G) = 0$ which is a contradiction. Thus $bp(G \odot K_m) = a_-(G \odot K_m)$, for $m = 2$ or $m = 3$. Moreover, if $m > 3$, then $a_-(G \odot K_m) > a_+(G \odot K_m)$. Therefore, $mn = a_-(G \odot K_m)$, for each $m > 1$. For each $i$, $\lambda_i + m - 1 + \sqrt{(m - 1 - \lambda_i)^2 + 4m} > 0$, we have $n$ positive eigenvalues. Then $\lambda_i + m - 1 - \sqrt{(m - 1 - \lambda_i)^2 + 4m}$
has to be a negative number for all $1 \leq i \leq n$. If $\lambda_i + m - 1 < 0$ for all $1 \leq i \leq n$, then it is clear that $\rho(G) < \frac{m}{m-1}$.

If $\lambda_i + m - 1 > 0$ for some $1 \leq i \leq n$, then $\lambda_i + m - 1 - \sqrt{(m - 1 - \lambda_i)^2 + 4m} < 0$ if and only if $\lambda_i < \frac{m}{m-1}$ for all $i : 1 \leq i \leq n$. Thus $\rho(G) < \frac{m}{m-1}$.

Now if $\rho(G) < \frac{m}{m-1}$, then $\lambda_i + m - 1 - \sqrt{(m - 1 - \lambda_i)^2 + 4m} < 0$ for all $1 \leq i \leq n$, so $a_-(G \circ K_m) = mn > a_+(G \circ K_m) = n$. Thus $bp(G \circ K_m) = mn = a_-(G \circ K_m) = \max\{a_+(G \circ K_m), a_-(G \circ K_m)\}$. Hence, $G \circ K_m$ is an eigensharp graph.

Next, we discuss the graph $K_m \circ G$. We specify a condition on $G$ that leads $K_m \circ G$ to be almost eigensharp.

To begin with, we present the following theorem for the biclique partition number of $K_m \circ G$ in general.

**Theorem 2.5.** For the graph $K_m \circ G$, $bp(K_m \circ G) = m(bp(G) + 1)$ for $m \geq 1$.

**Proof.** Let $G$ be a graph of order $n$. Then, $K_m \circ G$ consists of $K_m$ and $m$ copies of $G$; namely, $G_i : 1 \leq i \leq m$. Now, in order to define the biclique partition number of $K_m \circ G$, the values $bp(G)$ and $bp(K_m)$ must be taken into account. Let $V(K_m) = \{u_1, u_2, ..., u_m\}$ be the set of vertex of $K_m$. Then, $bp(K_m) = m - 1$. Let $S_i$ be the star in $K_m$ generated by $u_i$. Then, without loss of generality, we have $E(K_m) = \cup_{i=1}^{m-1} S_i$. Also, since $\{G_i\}_{i=1}^n$ is a family of disjoint graphs with $bp(G_i) = bp(G)$ for each $i$, we have $bp(K_m \circ G) \geq (m-1) + mbp(G)$. Now, since exceeding $u_m$ will result in the edges connecting $u_m$ and $G_m$ are not being covered, $S_m$; the star in $K_m$ generated by $u_m$, must be included in the partition. Hence $bp(K_m \circ G) = [(m - 1) + mbp(G)] + 1 = m(bp(G) + 1)$. □

**Theorem 2.6.** If $G$ is an $r$-regular eigensharp graph with $bp(G) = a_-(G)$, then $K_m \circ G$ is an eigensharp graph when $r = 1$; otherwise, it is an almost eigensharp graph for each $m > 1$.

**Proof.** Let $G$ be a graph of order $n$. Assume that $\lambda_i, 1 \leq i \leq n$ are the eigenvalues of $A(G)$ with $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = r$. Then, by Theorem 1.1

$$\sigma(K_m \circ G) = \left( (\lambda_1)^m, (\lambda_2)^m, ..., (\lambda_{n-1})^m, \left( \frac{1}{2} \left( -1 + r \pm \sqrt{(r+1)^2 + 4m} \right) \right)^m \right)^{m-1}.$$ 

$$\left( \frac{1}{2} \left( (m-1) + r \pm \sqrt{(r - (m-1))^2 + 4m} \right) \right)^1.$$
Suppose that \( t = a_+(G) < k = a_-(G) \) and \( bp(G) = k \). By Theorem 2.5, \( bp(K_m \circ G) = m(k + 1) \).

If \( r = 1 \), then \( (m - 1) + r + \sqrt{(r - (m - 1))^2 + 4m} > 0 \) and \( -1 + r + \sqrt{(r + 1)^2 + 4m} > 0 \). Also, \( (m - 1) + r - \sqrt{(r - (m - 1))^2 + 4m} < 0 \) and \( -1 + r - \sqrt{(r + 1)^2 + 4m} < 0 \). So \( a_-(K_m \circ G) = m + mk \) and \( a_+(K_m \circ G) = m(t-1)+m \). Then, \( bp(K_m \circ G) = m(k+1) = \max\{a_+(K_m \circ G), a_-(K_m \circ G)\} \). Therefore, \( K_m \circ G \) is an eigensharp graph.

If \( r > 1 \), then we have \( (m - 1) + r + \sqrt{(r - (m - 1))^2 + 4m} > 0 \), \( (m - 1) + r - \sqrt{(r - (m - 1))^2 + 4m} > 0 \), \(-1 + r + \sqrt{(r + 1)^2 + 4m} < 0 \), and \(-1 + r - \sqrt{(r + 1)^2 + 4m} < 0 \). So \( a_-(K_m \circ G) = m - 1 + mk \) and \( a_+(K_m \circ G) = m(t-1)+m+1 \). Then, \( bp(K_m \circ G) = m(k+1) = \max\{a_+(K_m \circ G), a_-(K_m \circ G)\} + 1 \). Hence \( K_m \circ G \) is an almost eigensharp graph. \( \square \)

### 3 The Corona Product Graphs With \( K_{n,n} \)

In this section, we study the eigensharp property of the corona product of a graph \( G \) of order \( m \) with a complete bipartite graph \( K_{n,n} \) and the corona product of a complete bipartite graph \( K_{n,n} \) with a regular graph \( G \) of order \( m \).

**Theorem 3.1.** Let \( G \) be a graph of order \( m \). Then \( bp(G \circ K_{n,n}) = 2|G| \).

**Proof.** Let \( G \) be a graph of order \( m \) and let \( V(G) = \{u_1, u_2, ..., u_m\} \) be the vertex set of \( G \). Then, \( G \circ K_{n,n} \) is formed by a copy of \( G \) and \( m \) copies of \( K_{n,n} \); namely, \( K_{n,n}^i : 1 \leq i \leq m \), such that \( u_i \) in \( G \) is adjacent to each vertex in \( K_{n,n}^i \).

In \( G \circ K_{n,n} \), \( \{K_{n,n}^i\}_{i=1}^m \) is a family of disjoint bicliques. So \( \{u_i \circ K_{n,n}^i\}_{i=1}^m \) is a family of disjoint subgraphs with \( bp(u_i \circ K_{n,n}^i) = 2 \), for each \( 1 \leq i \leq m \). In fact, the biclique partition covering of \( u_i \circ K_{n,n}^i \) with minimum cardinality is the star \( S_{u_i} \) generated by \( u_i \) with a biclique \( K_{n,n}^i \) for \( 1 \leq i \leq m \). This implies that the biclique partition of minimum cardinality of \( G \circ K_{n,n} \) is a union of all families of stars \( \{S_{u_i}\}_{i=1}^m \) with the family \( \{K_{n,n}^i\}_{i=1}^m \). Therefore, we have \( bp(G \circ K_{n,n}) = 2m = 2|G| \). \( \square \)

**Theorem 3.2.** Let \( G \) be a graph of order \( m \). Then the graph \( G \circ K_{n,n} \) is eigensharp if and only if \( \rho(G) < 2 \) for each \( n > 1 \).
Proof. Let $G$ be a graph of order $m$ and let $\lambda_i, 1 \leq i \leq m$ be the eigenvalues of $A(G)$. Then, by Theorem 1.1, we have
\[
\sigma(G \odot K_{n,n}) = \left( (−n)^m, (0)^{2m(n−1)}, \left( \frac{1}{2} \left( \lambda_i + n \pm \sqrt{(n-\lambda_i)^2 + 8n} \right) \right)^1 \right).
\]

By Theorem 3.1, $bp(G \odot K_{n,n}) = 2 |G|$. Assume that $G \odot K_{n,n}$ is an eigensharp graph. Then $2 |G| = \max\{a_+(G \odot K_{n,n}), a_-(G \odot K_{n,n})\}$. If $bp(G \odot K_{n,n}) = a_+(G \odot K_{n,n}) = 2 |G|$, then $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n} > 0$ for all $1 \leq i \leq m$ which implies that $\lambda_i > 2$. This means that $a_-(G) = 0$ which is a contradiction. Thus $bp(G \odot K_{n,n}) = 2 |G| = a_-(G \odot K_{m})$, so $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n}$ has to be negative number for all $1 \leq i \leq m$. Now, if $\lambda_i + n < 0$ then it is clear that $\lambda_i < 2$ for all $1 \leq i \leq m$. If $\lambda_i + n > 0$ for some $1 \leq i \leq m$, then the inequality $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n} < 0$ is true if and only if $\lambda_i < 2$ for all $1 \leq i \leq m$, so $\rho(G) < 2$. Now, if $\rho(G) < 2$, then $\lambda_i + n - \sqrt{(n-\lambda_i)^2 + 8n} < 0$ for all $1 \leq i \leq m$. Thus, $bp(G \odot K_{n,n}) = 2m = a_-(G \odot K_{n,n}) = \max\{a_+(G \odot K_{n,n}), a_-(G \odot K_{n,n})\}$ which implies that $G \odot K_{n,n}$ is an eigensharp graph. \qed

**Theorem 3.3.** For the graph $K_{n,n} \odot G$, $bp(K_{n,n} \odot G) = 2n(bp(G) + 1)$.

**Proof.** Let $V(K_{n,n}) = \{u_1, u_2, ..., u_{2n}\}$ be the vertex set of $K_{n,n}$. Then, $K_{n,n} \odot G$ is formed by a copy of $K_{n,n}$ and $2n$ copies of $G^i : 1 \leq i \leq 2n$, such that each vertex $u_i$ in $K_{n,n}$ is adjacent to each vertex in the $i$-th copy $G_i : 1 \leq i \leq 2n$. So, we have a family of disjoint subgraph; namely, $\{u_i \odot G_i\}_{i=1}^{2n}$. Thus, $bp(u_i \odot G_i) = 1 + bp(G_i)$ where the biclique partition covering of $u_i \odot G_i$, with minimum cardinality being a star generated by $u_i : 1 \leq i \leq 2n$ and a biclique partition of minimum cardinality of $G_i$ (see [3]). This shows that the biclique partition of minimum cardinality of $K_{n,n} \odot G$ is a union of the family of stars generated by $\{u_i\}_{i=1}^{2n}$ with the biclique partition of minimum cardinality of $G_i : 1 \leq i \leq 2n$. Therefore, we have $bp(G \odot K_{n,n}) = 2n + 2bp(G)$. \qed

**Theorem 3.4.** Suppose $G$ is an $r$-regular eigensharp graph of order $m$ with $bp(G) = a_-(G)$.

(i) If $n < \frac{m}{r}$, then $K_{n,n} \odot G$ is an eigensharp graph.

(ii) If $n \geq \frac{m}{r}$, then $K_{n,n} \odot G$ is almost eigensharp graph.

**Proof.** Let $G$ be a graph of order $m$. Assume that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m = r$ are the eigenvalues of $A(G)$. Then, by Theorem 1.1,
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\[ \sigma(K_{n,n} \odot G) = \left( \left( \frac{1}{2} (n + r \pm \sqrt{(r - n)^2 + 4m}) \right), \left( \frac{1}{2} (r \pm \sqrt{r^2 + 4m}) \right)^{2n-2}, \left( \frac{1}{2} (n + r \pm \sqrt{(r + n)^2 + 4m}) \right) \right)^1, \lambda^2, \lambda^3, ..., \lambda^m. \]

Assume that \( t = a_+(G) < k = a_-(G) \) and \( bp(G) = k \). By Theorem 3.3, \( bp(K_{n,n} \odot G) = 2n(k+1) \).

i) If \( n < \frac{2m}{r} \), then \(-n+r-\sqrt{(r+n)^2+4m} < 0, n+r-\sqrt{(r-n)^2+4m} < 0 \) and \( r-\sqrt{r^2+4m} < 0 \). Also, \(-n+r+\sqrt{(r+n)^2+4m} > 0, n+r+\sqrt{(r-n)^2+4m} > 0 \) and \( r+\sqrt{r^2+4m} > 0 \). So, \( a-(K_{n,n} \odot G) = 2nk+2n \) and \( a+(K_{n,n} \odot G) = 2n+2n(t-1) = 2nt \). Therefore, \( bp(K_{n,n} \odot G) = \max \{a-(K_{n,n} \odot G), a+(K_{n,n} \odot G)\} = 2n(k+1) \). Hence the graph \( K_{n,n} \odot G \) is eigensharp.

ii) If \( n \geq \frac{2m}{r} \), then \(-n+r-\sqrt{(r+n)^2+4m} < 0, n+r-\sqrt{(r-n)^2+4m} < 0 \) and \( r-\sqrt{r^2+4m} < 0 \). Also, \( n+r-\sqrt{(r-n)^2+4m} \geq 0, -n+r+\sqrt{(r+n)^2+4m} > 0, n+r+\sqrt{(r-n)^2+4m} > 0 \) and \( r+\sqrt{r^2+4m} > 0 \). So \( a-(K_{n,n} \odot G) = 2nk+2n-1 \) and \( a+(K_{n,n} \odot G) = 2n+2n(t-1) + 1 = 2nt + 1 \). Thus, \( \max \{a-(K_{n,n} \odot G), a+(K_{n,n} \odot G)\} = 2n(k+1) - 1 \). Therefore, \( bp(K_{n,n} \odot G) = \max \{a-(K_{n,n} \odot G), a+(K_{n,n} \odot G)\} + 1 \) and hence the graph \( K_{n,n} \odot G \) is almost eigensharp.

**Acknowledgment.** The third author is supported by the Scientific Research and Graduate Studies at Yarmouk University.

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