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Some results on traces of the generalized products and sums of positive semidefinite matrices

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Abstract

Matrix inequalities that expand certain scalar ones have been within the center of numerous researchers considerations. The purpose of this article is to prove the trace inequality depending on positive semidefinite block matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$. In this direction, we give some examples in support of the given concepts and presented results.

1 Introduction

In the last decade, various inequalities were proven with discoverers names attached to them. As a result, these inequalities played a role in the fundamental knowledge of analytical mathematics. Matrix inequalities arise in different branches of mathematics and science such as control theory, operations research, mathematical physics, approximation theory, financial matters, and the design of the strong controllers for linear dynamic systems (for details, see [1, 2, 4, 5, 7, 6, 8]).

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In this section, M_n stands for the space of $n \times n$ complex matrices and I_n denotes the identity matrix in M_n.

Definition 1.1. *[2] If $A = (a_{ij}) \in M_n$, then the trace of A, denoted by $tr(A)$ is defined as $tr(A) = \sum_{i=1}^{n} a_{ii}$*

Definition 1.2. *[2] A matrix $A \in M_n$ is said to be Hermitian (or self-adjoint) if $A^* = A$.*

Definition 1.3. *[2] A Hermitian matrix $A \in M_n$ is called positive semidefinite, and is denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in C^n$.*

Definition 1.4. *[2] If $A \in M_n$, then the absolute value of A is the matrix $|A| = (A^*A)^{\frac{1}{2}}$.*

Remark 1.5. *[2] The sum of two positive semidefinite matrices is positive semidefinite.*

Remark 1.6. *[2] The trace of a positive semidefinite matrix is a nonnegative real number.*

Remark 1.7. *(Arithmetic-Geometric Mean Inequalities)[1]*
$(\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i$, *where $x_i's$ are nonnegative real numbers and n is a positive integer.*

Theorem 1.8. *[2] A matrix $A \in M_n$ is positive semidefinite iff $A = B^*B$ for some $B \in M_n$.*

Theorem 1.9. *[2] Let $A \in M_n$ be positive semidefinite matrix. Then $trA^p \leq (trA)^p$, $p \geq 1$.*

Theorem 1.10. *[1] Let $A, B \in M_n$ be positive semidefinite matrices. Then $0 \leq tr(AB) \leq tr(A) \cdot tr(B)$.*

Theorem 1.11. *[8] Let $A, B, C \in M_n$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then $| \; tr(B^p) \; |^2 \leq (tr(| \; B \; |^p))^2 \leq tr(A^p) \cdot tr(C^p)$, for every integer $p \geq 1$. If $tr(B) \geq 0$ is a real number, then*

$$(tr(B^p))^2 \leq (tr(| \; B \; |^p))^2 \leq tr(A^p) \cdot tr(C^p).$$

2 Main Results

Theorem 2.1. *If $A, B \in M_n$ are positive semidefinite matrices, then* $(tr(A+B)^p)^2 \leq (tr(\mid A+B \mid^p))^2 \leq tr(I_n + A^2)^p tr(I_n + B^2)^p, p \geq 1.$

Proof. By Theorem 1.8, we get

$$\begin{bmatrix} I_n + B^*B & A^* + B^* \\ A+B & I_n + AA^* \end{bmatrix} = \begin{bmatrix} I_n & A^* \\ B & I_n \end{bmatrix}^* \begin{bmatrix} I_n & A^* \\ B & I_n \end{bmatrix} \geq 0.$$

Now applying Theorem 1.11 and $A, B \geq 0$, we have

$$\begin{aligned} (tr(A+B)^p)^2 &\leq (tr \mid A+B \mid^p)^2 \\ &\leq tr(I_n + A^2)^p tr(I_n + B^2)^p, p \geq 1. \end{aligned}$$

Hence

$$\begin{aligned} tr(A+B)^p &\leq [tr(I_n + A^2)^p tr(I_n + B^2)^p]^{\frac{1}{2}}, p \geq 1, \\ &\leq \frac{1}{2}[tr(I_n + A^2)^p + tr(I_n + B^2)^p], \ by\ Remark\ 1.7, \\ &\leq \frac{1}{2}[[tr(I_n + A^2)]^p + [tr(I_n + B^2)]^p], \ by\ Theorem\ 1.9. \end{aligned}$$

□

Corollary 2.2. *If $A, B \in M_n$ are positive semidefinite matrices, then*

$$\begin{aligned} tr(A+B) &\leq [tr(I_n + A^2) tr(I_n + B^2)]^{\frac{1}{2}} \\ &\leq \frac{1}{2}[tr(I_n + A^2) + tr(I_n + B^2)], \ by\ Remark\ 1.7, \\ &= \frac{1}{2}[2n + tr(A^2) + tr(B^2)], \\ &= n + \frac{1}{2}[tr(A^2) + tr(B^2)], \\ &\leq n + \frac{1}{2}[(tr(A))^2 + (tr(B))^2], \ by\ Theorem\ 1.9. \end{aligned}$$

Example 2.1. *Consider the positive semidefinite matrices* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. *Then*

$(A+B)^p = \begin{bmatrix} 3^p & 0 \\ 0 & 3^p \end{bmatrix}$ *and so* $(tr(A+B)^p)^2 = 4 \times 3^{2p}$.

$I_2 + A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *and so* $tr(I_2 + A^2)^p = 2^{p+1}$.

$I_2 + B^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ *and so* $tr(I_2 + B^2)^p = 2 \times 5^p$.

Proof. We show that $4 \times 3^{2p} \leq 2^{p+2}5^p$ using Mathematical Induction. When $p = 1$, we have $4 \times 3^2 \leq 2^3 \times 5$. Assume the statement is true for $p = k \in N$; i.e., $4 \times 3^{2k} \leq 2^{k+2}5^k$. We show that it is true for $p = k+1$; i.e., $4 \times 3^{2k+2} \leq 2^{k+3}5^{k+1}$:

$$\begin{aligned} 4 \times 3^{2k+2} &= 4 \times 3^{2k}9 \\ &\leq 2^{k+2}5^k9 \\ &= 36 \times 2^k5^k \\ &\leq 40 \times 2^k5^k \\ &= 2^{k+3}5^{k+1}. \end{aligned}$$

□

Example 2.2. *Consider the positive semidefinite matrices* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. *Then*

$(A+B)^p = \begin{bmatrix} 1 & 0 \\ 0 & 2^p \end{bmatrix}$ *and so* $(tr(A+B)^p)^2 = (2^p+1)^2$.

$I_2 + A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ *and so* $tr(I_2 + A^2)^p = 2^p + 1$.

$I_2 + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ *and so* $tr(I_2 + B^2)^p = 5^p + 1$.

It is clear that $(2^p+1)^2 \leq (2^p+1)(5^p+1)$.

Theorem 2.3. *If $A, B \in M_n$ are positive semidefinite matrices, then* $(tr(AB)^p)^2 \leq (tr \mid AB \mid^p)^2 \leq tr(I_n + A^2)^p tr(B^2)^p, p \geq 1.$

Proof. By Theorem 1.8, we get

$$\begin{bmatrix} I_n + A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} = \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix}^* \begin{bmatrix} A & B \\ I_n & 0 \end{bmatrix} \geq 0.$$

Now applying Theorem 1.11 and $A, B \geq 0$, we have

$$\begin{aligned} (tr(AB)^p)^2 &\leq (tr \mid AB \mid^p)^2 \\ &\leq tr(I_n + A^2)^p tr(B^2)^p, p \geq 1. \end{aligned}$$

Hence

$$\begin{aligned} tr(AB)^p &\leq [tr(I_n + A^2)^p tr(B^2)^p]^{\frac{1}{2}}, p \geq 1, \\ &\leq \frac{1}{2}[tr(I_n + A^2)^p + tr(B^2)^p], \ by\ Remark\ 1.7, \\ &\leq \frac{1}{2}[[tr(I_n + A^2)]^p + [tr(B^2)]^p], \ by\ Theorem\ 1.9. \end{aligned}$$

□

Corollary 2.4. *If $A, B \in M_n$ are positive semidefinite matrices, then*

$$\begin{aligned} tr(AB) &\leq [tr(I_n + A^2) tr B^2]^{\frac{1}{2}} \\ &\leq \frac{1}{2}[tr(I_n + A^2) + tr B^2], \ by\ Remark\ 1.7, \\ &= \frac{1}{2}[n + tr A^2 + tr B^2] \\ &\leq \frac{1}{2}[n + (tr A)^2 + (tr B)^2], \ by\ Theorem\ 1.9. \end{aligned}$$

Example 2.3. *Consider the positive semidefinite matrices* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. *Then*

$(AB)^p = \begin{bmatrix} 2^p & 0 \\ 0 & 2^p \end{bmatrix}$ *and so* $(tr(AB)^p)^2 = 2^{2p+2}$.

$I_2 + A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *and so* $tr(I_2 + A^2)^p = 2^{p+1}$.

$B^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ *and so* $tr(B^2)^p = 2^{2p+1}$.

It is clear that $2^{2p+2} \leq 2^{p+1} 2^{2p+1} = 2^{3p+2}$.

Example 2.4. *Consider the matrices* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ *are positive semidefinite matrices. Then*
$(AB)^p = \begin{bmatrix} 2^p & 0 \\ 0 & 0 \end{bmatrix}$ *and so* $(tr(AB)^p)^2 = 2^{2p}$.

$I_2 + A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ *and so* $tr(I_2 + A^2)^p = 2^p + 1$.

$B^2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ *and so* $tr(B^2)^p = 4^p$.
It is clear that $4^p \leq (2^p + 1)4^p$.

3 Conclusion and Open Question

In this study, we have proved that the trace inequalities depending on the positive semidefinite block matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ are $tr(A+B) \leq \frac{1}{2}[2n + tr(A^2) + tr(B^2)]$ and $tr(AB) \leq \frac{1}{2}[n + trA^2 + trB^2]$.
In addition, we would like to bring the researchers attention to the following question:
Can one give all cases for $tr(A + B)$ and $tr(AB)$?

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