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Three combined sequences related to k-Fibonacci sequences

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Abstract

In this paper, we introduce and investigate three new combined sequences related to k-Fibonacci sequences.

1 Introduction

For any positive number k, Falcon and Plaza [1] defined k-Fibonacci sequences $\{F_{k,n}\}_{n\in\mathbf{N}}$ recurrently by:

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1,$$

where $F_{k,0} = 0$ and $F_{k,1} = 1$.

In 2018, Atanassov [2] studied two new combined 3-Fibonacci sequences and later in the same year, he studied two additional new combined 3-Fibonacci sequences [3].

In 2021, Nubpetchploy and Pakapongpun [4] generated three combined sequences related to Jacobsthal sequences.

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In this paper, we introduce three new sequences and three new combined sequences related to k-Fibonacci sequences.

2 Main results

Let a, b, c and d be arbitrary real numbers. Our first sequence has the form

$$\begin{aligned} \gamma_{n+2} &= k\gamma_{n+1} + \gamma_n \\ \alpha_{n+1} &= k\gamma_{n+1} + \beta_n \\ \beta_{n+1} &= k\gamma_{n+1} + \alpha_n, \end{aligned}$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c$ and $\gamma_1 = d$ for integers $n \geq 0$.

The first few members of the sequences $\{F_{k,n}\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ with respect to n are represented in Table 1. Moreover, Tables 2 and 3 show the first few members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, respectively.

Table 1

n	$\{F_{k,n}\}_{n=0}^{\infty}$	$\{\gamma_n\}_{n=0}^{\infty}$
0	0	c
1	1	d
2	k	$kd + c$
3	$k^2 + 1$	$k^2d + kc + d$
4	$k^3 + 2k$	$k^3d + k^2c + 2kd + c$
5	$k^4 + 3k^2 + 1$	$k^4d + k^3c + 3k^2d + 2kc + d$
6	$k^5 + 4k^3 + 3k$	$k^5d + k^4c + 4k^3d + 3k^2c + 3kd + c$
7	$k^6 + 5k^4 + 6k^2 + 1$	$k^6d + k^5c + 5k^4d + 4k^3c + 6k^2d + 3kc + d$
8	$k^7 + 6k^5 + 10k^3 + 4k$	$k^7d + k^6c + 6k^5d + 5k^4c + 10k^3d + 6k^2c + 4kd + c$

Table 2	
n	$\{\alpha_n\}_{n=0}^{\infty}$
0	a
1	$kd+b$
2	$k^2d+k(c+d)+a$
3	$k^3d+k^2(c+d)+k(c+2d)+b$
4	$k^4d+k^3(c+d)+k^2(c+3d)+k(2c+2d)+a$
5	$k^5d+k^4(c+d)+k^3(c+4d)+k^2(3c+3d)+k(2c+3d)+b$
6	$k^6d+k^5(c+d)+k^4(c+5d)+k^3(4c+4d)+k^2(3c+6d)+k(3c+3d)+a$

Table 3	
n	$\{\beta_n\}_{n=0}^{\infty}$
0	b
1	$kd+a$
2	$k^2d+k(c+d)+b$
3	$k^3d+k^2(c+d)+k(c+2d)+a$
4	$k^4d+k^3(c+d)+k^2(c+3d)+k(2c+2d)+b$
5	$k^5d+k^4(c+d)+k^3(c+4d)+k^2(3c+3d)+k(2c+3d)+a$
6	$k^6d+k^5(c+d)+k^4(c+5d)+k^3(4c+4d)+k^2(3c+6d)+k(3c+3d)+b$

Theorem 2.1. *For each natural number with the elements of the k-Fibonacci sequences.*

(a) $\gamma_n = F_{k,n}d + F_{k,n-1}c$

(b) $\alpha_{2n} = (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + a$

(c) $\beta_{2n} = (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + b$

(d) $\alpha_{2n-1} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-2} + F_{k,2n-1} - 1)c + b$

(e) $\beta_{2n-1} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-2} + F_{k,2n-1} - 1)c + a.$

Proof.

(a) We will prove **(a)** by mathematical induction.
If $n = 1$, then $\gamma_1 = F_{k,1}d + F_{k,0}c = d$.
Assume the truth of the statements for some $n-1$ and n; that is,

$$\gamma_{n-1} = F_{k,n-1}d + F_{k,n-2}c$$

and

$$\gamma_n = F_{k,n}d + F_{k,n-1}c.$$

Now consider

$$\begin{aligned}\gamma_{n+1} &= k\gamma_n + \gamma_{n-1}\\ &= k(F_{k,n}d + F_{k,n-1}c) + F_{k,n-1}d + F_{k,n-2}c\\ &= (kF_{k,n} + F_{k,n-1})d + (kF_{k,n-1} + F_{k,n-2})c\\ &= F_{k,n+1}d + F_{k,n}c,\end{aligned}$$

which is the statement for $n + 1$. Therefore, the statement is true for all $n \geq 1$.

(b) We will prove **(b)** by mathematical induction as well.
If $n = 1$, then

$$\begin{aligned}\alpha_2 &= (F_{k,3} + F_{k,2} - 1)d + (F_{k,2} + F_{k,1} - 1)c + a\\ &= (k^2 + 1 + k - 1)d + (k + 1 - 1)c + a\\ &= (k^2 + k)d + kc + a = k^2 d + k(c + d) + a,\end{aligned}$$

which is true.
Assume the truth of the statements for some $n - 1$ and n; that is,

$$\alpha_{2n-2} = (F_{k,2n-1} + F_{k,2n-2} - 1)d + (F_{k,2n-2} + F_{k,2n-3} - 1)c + a$$

and

$$\alpha_{2n} = (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + a.$$

Now consider

$$\begin{aligned}\alpha_{2n+2} =& k\gamma_{2n+2} + \beta_{2n+1}\\ =& k(F_{k,2n+2}d + F_{k,2n+1}c) + k\gamma_{2n+1} + \alpha_{2n}\\ =& k(F_{k,2n+2}d + F_{k,2n+1}c) + k(F_{k,2n+1}d + F_{k,2n}c)\\ &+ (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + a\\ =& (kF_{k,2n+2} + F_{k,2n+1})d + (kF_{k,2n+1} + F_{k,2n})d\\ &+ (kF_{k,2n+1} + F_{k,2n})c + (kF_{k,2n} + F_{k,2n-1})c - d - c + a\\ =& (F_{k,2n+3} + F_{k,2n+2} - 1)d + (F_{k,2n+2} + F_{k,2n+1} - 1)c + a,\end{aligned}$$

which is the statement for $n + 1$. Therefore, the statement is true for all $n \geq 1$.

(c) The proof of **(c)** is similar to the proof of **(b)**.

(d) Consider,

$$\begin{aligned}\alpha_{2n+1} =&k\gamma_{2n+1} + \beta_{2n}\\ =&k(F_{k,2n+1}d + F_{k,2n}c) + (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + b\\ =&(kF_{k,2n+1} + F_{k,2n})d + (kF_{k,2n} + F_{k,2n-1})c + F_{k,2n+1}d + F_{k,2n}c - c - d + b\\ =&F_{k,2n+2}d + F_{k,2n+1}c + F_{k,2n+1}d + F_{k,2n}c - c - d + b\\ =&(F_{k,2n+2} + F_{k,2n+1} - 1)d + (F_{k,2n+1} + F_{k,2n} - 1)c + b.\end{aligned}$$

(e) The proof of **(e)** is similar to the proof of **(d)**. □

The second sequence has the form

$$\begin{aligned}\gamma_{n+2} &= k\gamma_{n+1} + \gamma_n\\ \alpha_{n+1} &= k\gamma_n + \beta_n\\ \beta_{n+1} &= k\gamma_n + \alpha_n\end{aligned}$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c$ and $\gamma_1 = d$, for integers $n \geq 0$.

The first few members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are represented in Tables 4 and 5, respectively.

Table 4

n	$\{\alpha_n\}_{n=0}^{\infty}$
0	a
1	$kc + b$
2	$k(c + d) + a$
3	$k^2d + k(2c + d) + b$
4	$k^3d + k^2(c + d) + k(2c + 2d) + a$
5	$k^4d + k^3(c + d) + k^2(c + 3d) + k(3c + 2d) + b$
6	$k^5d + k^4(c + d) + k^3(c + 4d) + k^2(3c + 3d) + k(3c + 3d) + a$
7	$k^6d + k^5(c + d) + k^4(c + 5d) + k^3(4c + 4d) + k^2(3c + 6d) + k(4c + 3d) + b$

Table 5

n	$\{\beta_n\}_{n=0}^{\infty}$
0	b
1	$kc+a$
2	$k(c+d)+b$
3	$k^2d+k(2c+d)+a$
4	$k^3d+k^2(c+d)+k(2c+2d)+b$
5	$k^4d+k^3(c+d)+k^2(c+3d)+k(3c+2d)+a$
6	$k^5d+k^4(c+d)+k^3(c+4d)+k^2(3c+3d)+k(3c+3d)+b$
7	$k^6d+k^5(c+d)+k^4(c+5d)+k^3(4c+4d)+k^2(3c+6d)+k(4c+3d)+a$

Theorem 2.2. *For each natural number,*

(a) $\gamma_n = F_{k,n}d + F_{k,n-1}c$

(b) $\alpha_{2n} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-1} + F_{k,2n-2} + F_{k,2} - 1)c + a$

(c) $\beta_{2n} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-1} + F_{k,2n-2} + F_{k,2} - 1)c + b$

(d) $\alpha_{2n-1} = (F_{k,2n-1} + F_{k,2n-2} - 1)d + (F_{k,2n-2} + F_{k,2n-3} + F_{k,2} - 1)c + b$

(e) $\beta_{2n-1} = (F_{k,2n-1} + F_{k,2n-2} - 1)d + (F_{k,2n-2} + F_{k,2n-3} + F_{k,2} - 1)c + a.$

Proof. The proofs are similar to theorem 2.1. □

The third sequence has the form

$$\begin{aligned}\gamma_{n+1} &= k\gamma_n + \frac{\alpha_n + \beta_n}{2}\\ \alpha_{n+1} &= k\gamma_n + \beta_n\\ \beta_{n+1} &= k\gamma_n + \alpha_n\end{aligned}$$

where $\alpha_0 = 2a, \beta_0 = 2b$ and $\gamma_0 = c$ for integers $n \geq 0$.

The first few members of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are in the table 6, 7 and table 8, respectively.

Theorem 2.3. *For each natural number,*

(a) $\gamma_{n+1} = \gamma_n(F_{k,2} + F_{k,1}) = \gamma_1(F_{k,2} + F_{k,1})^n$

Table 6	
n	$\{\alpha_n\}_{n=0}^{\infty}$
0	$2a$
1	$kc+2b$
2	$k^2c+k(a+b+c)+2a$
3	$k^3c+k^2(a+b+2c)+k(2a+2b+c)+2b$
4	$k^4c+k^3(a+b+3c)+k^2(3a+3b+3c)+k(3a+2b+c)+2a$
5	$k^5c+k^4(a+b+4c)+k^3(4a+4b+6c)+k^2(6a+6b+4c)+k(4a+4b+c)+2b$
6	$k^6c+k^5(a+b+5c)+k^4(5a+5b+10c)+k^3(10a+10b+10c)$ $+k^2(10a+10b+5c)+k(5a+5b+c)+2a$

Table 7	
n	$\{\beta_n\}_{n=0}^{\infty}$
0	$2b$
1	$kc+2a$
2	$k^2c+k(a+b+c)+2b$
3	$k^3c+k^2(a+b+2c)+k(2a+2b+c)+2a$
4	$k^4c+k^3(a+b+3c)+k^2(3a+3b+3c)+k(3a+2b+c)+2b$
5	$k^5c+k^4(a+b+4c)+k^3(4a+4b+6c)+k^2(6a+6b+4c)+k(4a+4b+c)+2a$
6	$k^6c+k^5(a+b+5c)+k^4(5a+5b+10c)+k^3(10a+10b+10c)$ $+k^2(10a+10b+5c)+k(5a+5b+c)+2b$

Table 8	
n	$\{\gamma_n\}_{n=0}^{\infty}$
0	c
1	$kc+a+b$
2	$k^2c+k(a+b+c)+a+b$
3	$k^3c+k^2(a+b+2c)+k(2a+2b+c)+a+b$
4	$k^4c+k^3(a+b+3c)+k^2(3a+3b+3c)+k(3a+2b+c)+a+b$
5	$k^5c+k^4(a+b+4c)+k^3(4a+4b+6c)+k^2(6a+6b+4c)+k(4a+4b+c)+a+b$
6	$k^6c+k^5(a+b+5c)+k^4(5a+5b+10c)+k^3(10a+10b+10c)$ $+k^2(10a+10b+5c)+k(5a+5b+c)+a+b$

(b) $\alpha_{2n}=\beta_{2n-1}=\gamma_1(F_{k,2}+F_{k,1})^{2n-1}+a-b$

(c) $\alpha_{2n-1}=\beta_{2n}=\gamma_1(F_{k,2}+F_{k,1})^{2n-2}+b-a.$

Proof.
(a) Since

$$\gamma_{n+1} = k\gamma_n + \frac{\alpha_n + \beta_n}{2}.$$

We know that

$$\frac{\alpha_n + \beta_n}{2} = k\gamma_{n-1} + \frac{\alpha_{n-1} + \beta_{n-1}}{2}.$$

Then

$$\gamma_{n+1} = k\gamma_n + k\gamma_{n-1} + \frac{\alpha_{n-1} + \beta_{n-1}}{2}$$

and

$$\frac{\alpha_{n-1} + \beta_{n-1}}{2} = \gamma_n - k\gamma_{n-1}.$$

Hence

$$\gamma_{n+1} = k\gamma_n + k\gamma_{n-1} + \gamma_n - k\gamma_{n-1} = \gamma_n(k+1) = \gamma_n(F_{k,2} + F_{k,1})$$

and

$$\begin{aligned}\gamma_{n+1} &= \gamma_n(F_{k,2} + F_{k,1}) \\ &= \gamma_{n-1}(F_{k,2} + F_{k,1})^2,\end{aligned}$$

we do the same thing and get

$$\gamma_{n+1} = \gamma_1(F_{k,2} + F_{k,1})^n.$$

(b) We will prove **(b)** by mathematical induction.
If $n = 1$, then

$$\begin{aligned}\alpha_2 &= \gamma_1(F_{k,2} + F_{k,1}) + a - b \\ &= (kc + a + b)(k+1) + a - b \\ &= k^2c + kc + ka + kb + a + b + a - b \\ &= k^2c + k(a + b + c) + 2a.\end{aligned}$$

Assume the statement is true for integers n; that is,

$$\alpha_{2n} = \gamma_1(F_{k,2} + F_{k,1})^{2n-1} + a - b.$$

Now consider

$$\begin{aligned}
\alpha_{2n+2} &= k\gamma_{2n+1} + \beta_{2n+1} \\
&= k\gamma_{2n+1} + k\gamma_{2n} + \alpha_{2n} \\
&= k\gamma_{2n+1} + k\gamma_{2n} + (k+1)^{2n-1}\gamma_1 + a - b \\
&= k(k+1)\gamma_{2n} + k\gamma_{2n} + (k+1)^{2n-1}\gamma_1 + a - b \\
&= k\gamma_{2n}(k+2) + (k+1)^{2n-1}\gamma_1 + a - b \\
&= (k^2+2k)\gamma_{2n} + (k+1)^{2n-1}\gamma_1 + a - b \\
&= (k^2+2k)(k+1)^{2n-1}\gamma_1 + (k+1)^{2n-1}\gamma_1 + a - b \\
&= (k+1)^{2n-1}\gamma_1(k^2+2k+1) + a - b \\
&= (k+1)^{2n+1}\gamma_1 + a - b,
\end{aligned}$$

which is the statement for $n+1$. Therefore the statement is true for all $n \geq 1$.

(c) The proof of **(c)** is similar to the proof of **(b)**. □

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References

[1] S. Falcon, A. Plaza, On the Fibonacci k-number, *Chaos, Solition and Fractals*, **32,** no. 5, (2007), 1615-=1664.

[2] K. T. Atanassov, On two new combined 3-Fibonacci sequences, *Note on Number Theory and Discrete Mathematics*, **24,** no. 2, (2018), 90–93.

[3] K. T. Atanassov, On two new combined 3-Fibonacci sequences part 2, *Note on Number Theory and Discrete Mathematics*, **24,** no. 3, (2018), 111–114.

[4] S. Nubpetchploy, A. Pakapongpun, Three combined sequences related to Jacobsthal sequences, *International Journal of Mathematics and Computer Science*, **16,** no. 1, (2021), 149–157.