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Complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle

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Abstract

The subject of this paper is complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle. Various properties of curvature tensor are investigated on the tangent bundle. We obtain a relation between scalar curvature tensors $\tilde{r}^C$ and r^C of a semi-symmetric metric P-connection $\tilde{\nabla}^C$ and the Levi-Civita connection ∇^C.

1 Introduction

In 1924, Friedmann and Schouten started the study of connections on a Riemannian manifold [10]. Later, the study was investigated further by Hayden [12], Pak [19] and Yano [22].
Consider a Riemannian manifold M of dimension n and let ∇ be the Levi-Civita connection with Riemannian metric g on M. A linear connection $\tilde{\nabla}$ is called symmetric connection if the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$

$$\tilde{T}(\gamma_1, \gamma_2) = \tilde{\nabla}_{\gamma_1}\gamma_2 - \tilde{\nabla}_{\gamma_2}\gamma_1 - [\gamma_1, \gamma_2], \tag{1}$$

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for all γ_1 and γ_2 on M, is zero on M, or else it is non-symmetric connection.

The linear connection $\tilde{\nabla}$ is called semi-symmetric if

$$\tilde{T}(\gamma_1, \gamma_2) = \pi(\gamma_2)\gamma_1 - \pi(\gamma_1)\gamma_2, \tag{2}$$
$$\pi(\gamma_1) = g(\gamma_1, P), \tag{3}$$

for all γ_1 and γ_2 on M and π is 1-form and P is a vector field [10].

Moreover, the linear connection $\tilde{\nabla}$ is called metric if $\tilde{\nabla}g = 0$, or else it is non-metric [5, 6, 7].

Recently, Chaubey et al. [2, 3, 4] defined a class of semi-symmetric metric connection on a Riemannian manifold and studied geometrical properties of it such as conformal, m-projective curvature tensors. However, the theory of lifts (complete and vertical) of tensor fields and connections was developed by Kabayashi and Yano [21]. The complete and vertical lifts of semi-symmetric non-metric and quarter-symmetric non-metric connections on Kähler manifold and an almost Hermitian manifold sudied by Khan [13, 14, 15]. Recently, several types of connection on tangent bundle have been studied [1, 11, 16, 17, 18]. The subject of this paper is to study complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle.

The main contributions are summarized as follows:

- Complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle.
- Various properties of curvature tensor are investigated on the tangent bundle.
- A relation between scalar curvature tensors $\tilde{r}^C$ and r^C of a semi-symmetric metric P-connection $\tilde{\nabla}^C$ and the Levi-Civita connection ∇^C is obtained.

2 Preliminaries

Let M be an n-dimensional differentiable manifold and let TM be its tangent bundle.

Lemma 2.1. *The vertical and complete lifts of a vector field, 1-form, tensor field of type (1,1) and affine connection ∇ are given by $\gamma_1^V, \pi^V, F^V, \nabla^V$ and*

$\gamma_1^C, \pi^C, F^C, \nabla^C$, *respectively [8, 15, 23].*

$$\begin{aligned}
\pi^V(\gamma_1^C) &= \pi^C(\gamma_1^V) = \pi(\gamma_1)^V, \pi^C(\gamma_1^C) = \pi(\gamma_1)^C, && (4)\\
F^V\gamma_1^C &= (F\gamma_1)^V, F^C\gamma_1^C = (F\gamma_1)^C, && (5)\\
[\gamma_1,\gamma_2]^V &= [\gamma_1^C,\gamma_2^V] = [\gamma_1^V,\gamma_2^C], [\gamma_1,\gamma_2]^C = [\gamma_1^C,\gamma_2^C], && (6)\\
\nabla^C_{\gamma_1^C}\gamma_2^C &= (\nabla_{\gamma_1}\gamma_2)^C, \nabla^C_{\gamma_1^C}\gamma_2^V = (\nabla_{\gamma_1}\gamma_2)^V. && (7)
\end{aligned}$$

2.1 Semi-symmetric non-metric P-connection

Let M be a Riemannian manifold of dimension n with Riemannian metric g. A linear connection $\tilde{\nabla}$ on M given by [2, 9]

$$\tilde{\nabla}_{\gamma_1}\gamma_2 = \nabla_{\gamma_1}\gamma_2 + \pi(\gamma_2)\gamma_1 - g(\gamma_1,\gamma_2)P, \tag{8}$$

where ∇ is a Levi-Civita connection, γ_1, γ_2 vector fields and π 1-form on M.

The connection $\tilde{\nabla}$ satisfying equations (2), (8), $\tilde{\nabla}g = 0$ and $\tilde{\nabla}P = 0$ is called a semi-symmetric metric P-connection on M.

3 Complete lifts of a semi-symmetric metric P-connection on a Riemannian manifold to its tangent bundle

Let M be an n-dimensional Riemannian manifold with the Riemannian metric g and let TM be its tangent bundle. Then g^C is a Riemannian metric in TM. Taking complete lifts of equations (2), (3), (8), it follows that [20]

$$\begin{aligned}
\tilde{T}^C(\gamma_1^C,\gamma_2^C) &= \pi^C(\gamma_2^C)\gamma_1^V + \pi^V(\gamma_2^C)\gamma_1^C \\
&- \pi^C(\gamma_1^C)\gamma_2^V - \pi^V(\gamma_1^C)\gamma_2^C, && (9)\\
\pi^C(\gamma_1^C) &= g^C(\gamma_1^C, P^C). && (10)
\end{aligned}$$

A linear connection $\tilde{\nabla}^C$ defined by

$$\begin{aligned}
\tilde{\nabla}^C_{\gamma_1^C}\gamma_2^C &= \nabla^C_{\gamma_1^C}\gamma_2^C + \pi^C(\gamma_2^C)\gamma_1^V + \pi^V(\gamma_2^C)\gamma_1^C \\
&- g^C(\gamma_1^V,\gamma_2^C)P^C + g^C(\gamma_1^C,\gamma_2^C)P^V, && (11)
\end{aligned}$$

is said to be a semi-symmetric non-metric P-connection if the torsion tensor $\tilde{T}^C$ of TM with respect to $\tilde{\nabla}^C$ satisfies equations (9) and (10), the Riemannian metric $\tilde{\nabla}^C_{\gamma_1^C} g^C = 0$ and $\tilde{\nabla}^C_{\gamma_1^C} P^C = 0$.

As a consequence of equations (10), (11), $\tilde{\nabla}^C_{\gamma_1^C} g^C = 0$ and $\tilde{\nabla}^C_{\gamma_1^C} P^C = 0$, we have

$$\begin{aligned}\tilde{\nabla}^C_{\gamma_1^C} P^C = 0 \Longleftrightarrow \tilde{\nabla}^C_{\gamma_1^C} P^C &= \pi^C(\gamma_1^C)P^V + \pi^V(\gamma_1^C)P^C \\ &- \pi^C(P^C)\gamma_1^V + \pi^V(P^C)\gamma_1^C. \end{aligned} \tag{12}$$

and

$$\begin{aligned}(\tilde{\nabla}^C_{\gamma_1^C}\pi^C)(\gamma_2^C) &= (\nabla^C_{\gamma_1^C}\pi^C)(\gamma_2^C) \\ &+ \pi^C(P^C)g^C(\gamma_1^V, \gamma_2^C) + \pi^V(P^C)g^C(\gamma_1^C, \gamma_2^C) \\ &- \pi^V(\gamma_1^C)\pi^C(\gamma_2^C) - \pi^C(\gamma_1^C)\pi^V(\gamma_2^C), \end{aligned} \tag{13}$$

for all γ_1 and γ_2 on M.

Theorem 3.1. *Let M be n-dimensional Riemannian manifold with metric g and let TM be its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric P-connection $\tilde{\nabla}^C$. Then*

$$\begin{aligned}(i)\quad R^C(\gamma_1^C, \gamma_2^C)P^C &= \nabla^C_{\gamma_1^C}\{\pi^C(\gamma_2^C)P^V + \pi^V(\gamma_2^C)P^C\} \\ &- \nabla^C_{\gamma_2^C}\{\pi^C(\gamma_1^C)P^V + \pi^V(\gamma_1^C)P^C\} \\ &- \pi^C([\gamma_1, \gamma_2]^C)P^V - \pi^V([\gamma_1, \gamma_2]^C)P^C \\ &+ \pi^C(P^C)[\gamma_1, \gamma_2]^V + \pi^V(P^C)[\gamma_1, \gamma_2]^C \\ &= \pi^C(P^C)\pi^C(\gamma_1^C)\gamma_2^V + \pi^C(P^C)\pi^V(\gamma_1^C)\gamma_2^C \\ &+ \pi^V(P^C)\pi^C(\gamma_1^C)\gamma_2^C - \pi^C(P^C)\pi^C(\gamma_2^C)\gamma_1^V \\ &- \pi^C(P^C)\pi^V(\gamma_2^C)\gamma_1^C - \pi^V(P^C)\pi^C(\gamma_2^C)\gamma_1^C, \\ (ii)\quad R(P^C, \gamma_1^C)\gamma_2^C &= \pi^C(P^C)\pi^C(\gamma_2^C)\gamma_1^V + \pi^C(P^C)\pi^V(\gamma_2^C)\gamma_1^C \\ &+ \pi^V(P^C)\pi^C(\gamma_2^C)\gamma_1^C - \pi^C(P^C)g^C(\gamma_1^C, \gamma_2^C)P^V \\ &- \pi^C(P^C)g^C(\gamma_1^V, \gamma_2^C)P^C - \pi^V(P^C)g^C(\gamma_1^C, \gamma_2^C)P^C, \\ (iii)\quad \pi^C(R^C(\gamma_1^C, \gamma_2^C)\gamma_3^C) &= \pi^C(P^C)\pi^C(\gamma_2^C)g^C(\gamma_1^V, \gamma_3^C) + \pi^C(P^C)\pi^V(\gamma_2^C)g^C(\gamma_1^C, \gamma_3^C) \\ &+ \pi^V(P^C)\pi^C(\gamma_2^C)g^C(\gamma_1^C, \gamma_3^C) - \pi^C(P^C)\pi^C(\gamma_1^C)g^C(\gamma_2^V, \gamma_3^C) \\ &- \pi^C(P^C)\pi^V(\gamma_1^C)g^C(\gamma_2^C, \gamma_3^C) - \pi^V(P^C)\pi^C(\gamma_1^C)g^C(\gamma_2^C, \gamma_3^C), \end{aligned}$$

for all $\gamma_1, \gamma_2, \gamma_3$ on M.

Proof. The Riemannian curvature tensor R with ∇ is given by

$$R(\gamma_1, \gamma_2, \gamma_3) = \nabla_{\gamma_1}\nabla_{\gamma_2}\gamma_3 - \nabla_{\gamma_2}\nabla_{\gamma_1}\gamma_3 - \nabla_{[\gamma_1,\gamma_2]}\gamma_3 \tag{14}$$

for all $\gamma_1, \gamma_2, \gamma_3$ on M.

From (10), (12) and (13), the Riemannian curvature tensor R becomes

$$\begin{aligned} R^C(\gamma_1^C, \gamma_2^C)P^C &= \nabla^C_{\gamma_1^C}\{\pi^C(\gamma_2^C)P^V + \pi^V(\gamma_2^C)P^C\} \\ &- \nabla^C_{\gamma_2^C}\{\pi^C(\gamma_1^C)P^V + \pi^V(\gamma_1^C)P^C\} \\ &- \pi^C([\gamma_1,\gamma_2]^C)P^V - \pi^V([\gamma_1,\gamma_2]^C)P^C \\ &+ \pi^C(P^C)[\gamma_1,\gamma_2]^V + \pi^V(P^C)[\gamma_1,\gamma_2]^C \\ &= \pi^C(P^C)\pi^C(\gamma_1^C)\gamma_2^V + \pi^C(P^C)\pi^V(\gamma_1^C)\gamma_2^C + \pi^V(P^C)\pi^C(\gamma_1^C)\gamma_2^C \\ &- \pi^C(P^C)\pi^C(\gamma_2^C)\gamma_1^V - \pi^C(P^C)\pi^V(\gamma_2^C)\gamma_1^C - \pi^V(P^C)\pi^C(\gamma_2^C)\gamma_1^C. \end{aligned}$$

The inner product of above equation with γ_4 gives

$$\begin{aligned} 'R^C(\gamma_1^C, \gamma_2^C, P^C, \gamma_4^C) &= \pi^C(P^C)\pi^C(\gamma_1^C)g^C(\gamma_2^V, \gamma_4^C) + \pi^C(P^C)\pi^V(\gamma_1^C)g^C(\gamma_2^C, \gamma_4^C) \\ &+ \pi^V(P^C)\pi^C(\gamma_1^C)g^C(\gamma_2^C, \gamma_4^C) - \pi^C(P^C)\pi^C(\gamma_1^C)g^C(\gamma_2^V, \gamma_4^C) \\ &- \pi^C(P^C)\pi^V(\gamma_1^C)g^C(\gamma_2^C, \gamma_4^C) - \pi^V(P^C)\pi^C(\gamma_1^C)g^C(\gamma_2^C, \gamma_4^C). \end{aligned}$$

where $'R^C(\gamma_1^C, \gamma_2^C, P^C, \gamma_4^C) = gR(^C(\gamma_1^C, \gamma_2^C, P^C, \gamma_4^C))$.

The parts (ii) and (iii) of Theorem 3.1 are obtained by using symmetric properties of R and the above equation.

Theorem 3.2. *Let M be an n-dimensional Riemannian manifold with metric g and let TM be its tangent bundle with Riemannian metric g^C endowed with a semi-symmetric non-metric P-connection $\tilde{\nabla}^C$. Then a relation between the scalar curvatures $\tilde{r}$ and r is given by*

$$\tilde{r} = r + (n-1)\pi^C(P^C), \quad \tilde{r}^C = \tilde{r}, r^C = r. \tag{15}$$

Proof. Let $\tilde{R}$ be the curvature tensor corresponding to a semi-symmetric non-metric P-connection $\tilde{\nabla}$ and related to R corresponding to ∇ as

$$\begin{aligned} \tilde{R}^C(\gamma_1^C, \gamma_2^C)\gamma_3^C &= R^C(\gamma_1^C, \gamma_2^C)\gamma_3^C - \Phi^C(\gamma_2^C, \gamma_3^C)\gamma_1^C + \Phi^C(\gamma_1^C, \gamma_3^C)\gamma_2^C \quad (16) \\ &- g^C(\gamma_2^V, \gamma_3^C)(A\gamma_1)^C - g^C(\gamma_2^C, \gamma_3^C)(A\gamma_1)^V \quad (17) \\ &+ g^C(\gamma_1^V, \gamma_3^C)(A\gamma_2)^C - g^C(\gamma_1^C, \gamma_3^C)(A\gamma_2)^V, \quad (18) \end{aligned}$$

where Φ is a tensor of type (0,2) given by

$$\begin{aligned} \Phi^C(\gamma_1^C, \gamma_2^C) = g^C(A\gamma_1, \gamma_2)^C &= (\nabla^C_{\gamma_1^C}\pi^C)(\gamma_2^C) - \pi^C(\gamma_1^C)\pi^V(\gamma_1^C) - \pi^V(\gamma_1^C)\pi^C(\gamma_1^C) \\ &+ \frac{1}{2}(\pi^C(P^C)g^C(\gamma_1^V, \gamma_2^C) + \pi^V(P^C)g^C(\gamma_1^C, \gamma_2^C)) \quad (19) \end{aligned}$$

which is equivalent to

$$(A\gamma_1)^C = (\nabla_{\gamma_1} P)^C - \pi^C(\gamma_1^C)P^V - \pi^V(\gamma_1^C)P^C + \frac{1}{2}(\pi^C(P^C)\gamma_1^V + \pi^V(P^C)\gamma_1^C) \tag{20}$$

Making use of (10) and (12), (19) and (20) become

$$\Phi^C(\gamma_1^C, \gamma_2^C) = -\frac{1}{2}(\pi^C(P^C)g^C(\gamma_1^V, \gamma_2^C) + \pi^V(P^C)g^C(\gamma_1^C, \gamma_2^C)) \tag{21}$$

and

$$(A\gamma_1)^C = -\frac{1}{2}(\pi^C(P^C)\gamma_1^V - \pi^V(P^C)\gamma_1^C). \tag{22}$$

Using (21) and (22) in (16), the obtained equation is

$$\begin{aligned} \tilde{R}^C(\gamma_1^C, \gamma_2^C)\gamma_3^C &= R^C(\gamma_1^C, \gamma_2^C)\gamma_3^C + \pi^C(P^C)g^C(\gamma_2^V, \gamma_3^C)\gamma_1^C & (23) \\ &+ \pi^C(P^C)g^C(\gamma_2^C, \gamma_3^C)\gamma_1^V + \pi^V(P^C)g^C(\gamma_2^C, \gamma_3^C)\gamma_1^C & (24) \\ &- \pi^C(P^C)g^C(\gamma_1^V, \gamma_3^C)\gamma_2^C - \pi^C(P^C)g^C(\gamma_1^C, \gamma_3^C)\gamma_2^V & (25) \\ &- \pi^V(P^C)g^C(\gamma_1^C, \gamma_3^C)\gamma_2^C. & (26) \end{aligned}$$

Contracting (23) along γ_1^C, the obtained equation is

$$\tilde{S}^C(\gamma_2^C, \gamma_3^C) = S^C(\gamma_2^C, \gamma_3^C) + (n-1)\{\pi^C(P^C)g^C(\gamma_2^V, \gamma_3^C) + \pi^V(P^C)g^C(\gamma_2^C, \gamma_3^C)\}. \tag{27}$$

equivalent to

$$(\tilde{Q}\gamma_2)^C = (Q\gamma_2)^C + (n-1)\{\pi^C(P^C)\gamma_2^V + \pi^V(P^C)\gamma_2^C\}, \tag{28}$$

for all vector fields γ_2^C and γ_3^C on TM. Here $\tilde{Q}^C$ and Q^C are the complete lift of Ricci operators corresponding to the Ricci tensors $\tilde{Q}^C$ and Q^C complete lifts $\tilde{S}^C$ and S^C Ricci tensors $\tilde{S}$ and S of the connections $\tilde{\nabla}^C$ and ∇^C, respectively; that is, $\tilde{S}^C(\gamma_2^C, \gamma_3^C) = g^C(\tilde{Q}^C\gamma_2^C, \gamma_3^C)$ and $S^C(\gamma_2^C, \gamma_3^C) = g^C(Q^C\gamma_2^C, \gamma_3^C)$ Again contracting (28) along γ_2^C gives

$$\tilde{r} = r + (n-1)\pi^C(P^C), \quad \tilde{r}^C = \tilde{r}, r^C = r, \tag{29}$$

where $\tilde{r}$ and r denote the scalar curvatures corresponding to the semi-symmetric non-metric P-connection $\tilde{\nabla}^C$ and the Levi-Civita connection ∇^C, respectively.

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