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Principally Pure Submodules

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Abstract

In this paper, two concepts are introduced: Principally pure submodules, and principally regular modules. Also, we have studied the relationship between principally pure and principally injective. We prove that a module M is principally pure in any principally injective module containing it as a submodule if and only if M is a principally injective module.

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1 Introduction

In this paper, R is an associative ring with identity, all homomorphism are R-homomorphisms, all modules are unitary left R-modules. A submodule M of a module N is called a *pure* submodule [1] if for any commutative diagram with a finitely generated submodule I of a free module F

$$\begin{array}{ccc} I & \hookrightarrow & F \\ \downarrow & \swarrow & \downarrow \\ M & \hookrightarrow & N \end{array} \qquad (1)$$

there is an R-homomorphism $F \longrightarrow M$ making the upper triangle commutative. A module M is absolutely pure if for any finitely generated submodule I of a free module F, any homomorphism $I \to M$ can be extended to a homomorphism $F \to M$ [2, p.300]. A module M is said to be injective (resp., principally injective, absolutely self pure, absolutely neat, absolutely self neat, principally self injective) if for each homomorphism $f : I \to M$, where I is left ideal (resp., principal left ideal, finitely generated left ideal with $\ker f \in \bar{\Omega}(M)$, maximal left ideal, maximal left ideal with $\ker f \in \bar{\Omega}(M)$, principal left ideal with $\ker f \in \bar{\Omega}(M)$) of R, there exists a homomorphism $g : R \to M$ extending f, see [2, p.131], [3, p.96], and [1, 4, 5]. Here by $\bar{\Omega}(M)$ we mean the filter generated by $\Omega(M)$ in the lattice of left ideals of R where $\Omega(M)$ is the set of all left ideales I of R such that $I \supseteq \mathrm{ann}(m)$ for some $m \in M$. Each of the concepts mentioned above fulfills an extension property. Equivalently, a module M is injective (resp., absolutely pure, absolutely self pure, absolutely neat, absolutely self neat, principally self injective) if it is direct summand (resp., pure, self pure, neat, self neat, principally self pure) in each module containing it as a submodule see [2, p.130], [2, p.301], and [1, 4, 5]. Here, we give a description of principally injective modules as a submodule of other modules analogous to the above characterizations. We prove that a module is principally injective if and only if principally pure in every containing module. By using principally pure submodules Von Neumann regular is characterized.

2 Principally Pure Submodules

Definition 2.1. *A submodule L of a module N is called a principally pure submodule (or p-pure submodule for short) if for any commutative diagram with I a principal left ideal of R:*

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow f & \swarrow & \downarrow h \\ L & \hookrightarrow & N \end{array} \qquad (2)$$

There is a homomorphism $R \to L$ making the upper triangle commutative.

The following result proves some properties of p-pure submodules.

Proposition 2.2. *Let $L \subseteq M \subseteq N$ be modules.*
1. If L is p-pure in M and M is p-pure in N, then L is p-pure in N.
2. If L is p-pure in N, then L is p-pure in M.

Proof. 1. Consider the commutative diagram (2), with I being a principal left ideal of R. By considering f as a homomorphism $I \to M$ and since M is p-pure in N, we see, for the above diagram with L replaced by M, that there is a homomorphism $\beta : R \to M$ making the upper triangle commutative. So we have the commutative square:

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow f & \swarrow g & \downarrow \beta \\ L & \hookrightarrow & M \end{array} \qquad (3)$$

But L being p-pure in M gives the existence of a homomorphism $R \to L$ making the upper triangle commutative, hence we get the result.
2. Consider the commutative diagram (2) As M is a submodule in N, we can consider h as a homomorphism $R \to N$. By assumption, there is a homomorphism $g : R \to L$ making the upper triangle commutative. □

Here, we prove that a module is p-injective exactly when it is p-pure in every module containing it.

Theorem 2.3. *The following are equivalent, for a module M:*
(1) M is p-pure in any module containing it as a submodule.
(2) M is p-pure in any injective module containing it as a submodule.
(3) M is p-pure in its injective envelope.
(4) M is p-pure in any p-injective module containing it as a submodule.
(5) M is p-injective.

Proof. (1) ⇒ (4) ⇒ (2) ⇒ (3) are trivial.
(3) ⇒ (5) Let I be a principal left ideal of R, $f : I \to M$ a homomorphism, and $E = E(M)$ the injective envelope of M. By injectivity of E there exists a homomorphism $h : R \to E$ making diagram (2) commutative, with L and N being replaced by M and E respectively. By assumption, there is a homomorphism $R \to M$ making the upper triangle commutative. Thus, M is p-injective.
(5) ⇒ (1) Let $M \subseteq N$ be modules, I a principal left ideal of R, and $f : I \to M$ any homomorphism. Suppose there is a homomorphism $R \to N$ making the diagram (2), with L replaced by M, commutative. Since M is p-injective, there is a homomorphism $R \to M$ extending f and making the upper triangle commutative. Hence, M is p-pure in N. □

Examples 2.4.
(1) Every pure submodule is principally pure.
(2) Using Theorem 2.3, any p-injective module that is not absolutely pure is principally pure (but not pure) in its injective envelope [1].

Recall that a ring R is Von Neumann regular if any principal left ideal of R is a direct summand [2, p.21]. A module A is called *regular* module if every submodule of A is pure in A by [8]. The ring R is Von Neumann regular if and only if every R-module is regular by [2, p.316].

Definition 2.5. *An R-module M called* principally regular *(or p-regular for short) if every submodule of M is p-pure in M.*

Here, we show that regularity and p-regularity are equivalent properties for the ring R.

Proposition 2.6. *The following are equivalent for a ring R:*
(1) R is Von Neumann regular.
(2) R is p-regular.
(3) Every principal left ideal of R is p-pure in R.

Proof. (1) ⇒ (2) By using (1) of Examples 2.4 and [2, p.316].
(2) ⇒ (3) clear.
(3) ⇒ (1) For every principal left ideal I of R, there is a homomorphism $g : R \to I$, then in diagram (2), with M and N replaced by I and R respectively and f and h are the identity maps. This means that $R = I \oplus \ker g$. So, R is Von Neumann regular. □

Recall that a module M is called *pure Baer injective* module [6] if for each pure left ideal I of R, any homomorphism $I \to M$ can be extended to a homomorphism $R \to M$. In [7], Theorem 2. It is proved that every module can be embedded as a pure submodule of a pure Baer injective module.
Using the above concepts, we can characterize Von Neumann regular rings.

Theorem 2.7. *The following are equivalent for a ring R:*
(1) R is Von Neumann regular.
(2) For all modules $M \subseteq N$, we have M p-pure in N.
(3) Every principal left ideal of R is p-pure in R.
(4) Every principal left ideal of R is pure in R.
(5) Every pure Baer injective module is p-injective.
(6) All R-modules are p-injective.

Proof. (1) $\Rightarrow$ (2) Since N is regular, M must be pure in N, and so p-pure by Examples 2.4(1).
(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial.
(3) $\Rightarrow$ (4) Let I be a principal left ideal of R. If I p-pure in R, then in diagram (1), with M and N replaced by I and R respectively and f and h are the identity maps. There must exist a homomorphism $R \to I$ making the upper triangle commutative. Therefore, I is a direct summand of R, so it is pure in R.
(5) $\Rightarrow$ (6) By [7, Theorem 2] any module can be embedded as a pure submodule of a pure Baer injective, hence p-injective module. This means that every module M is p-)pure in some p-injective module N. But by Theorem 2.3, all p-injective modules are p-pure in their injective envelopes and since $E(M) \subseteq E(N)$, we see, by Proposition 2.2, that M is p-pure in $E(M)$. Hence M is p-injective.
(6) $\Rightarrow$ (1) A ring R is Von Neumann regular if and only if every R-module is p-injective [3, p. 96]. □

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