International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 723–730

Regularities of ordered semigroups in terms of (m, n)-ideals and n-interior ideals

Nuchanat Tiprachot[1], **Nareupanat Lekkoksung**[2],
Bundit Pibaljommee[3]

[1,3]Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, Thailand

[2]Division of Mathematics
Faculty of Engineering
Rajamangala University of Technology Isan
Khon Kaen Campus
Khon Kaen 40000, Thailand

email: nuchanatt@kkumail.com, nareupanat.le@rmuti.ac.th,
banpib@kku.ac.th

(Received October 3, 2021, Accepted November 17, 2021)

Abstract

We give the concept of n-interior ideals which is a generalization of interior ideals. Then we characterize thirteen regularities of ordered semigroups classified by linear inequations in terms of (m, n)-ideals and n-interior ideals.

1 Introduction

The characterizations of ordered semigroups can be formulated in terms of their ideals. Therefore, the ideal theory of ordered semigroups plays a vital

Key words and phrases: Regularities of ordered semigroup, (m, n)-ideal, n-interior ideal.
AMS (MOS) Subject Classifications: 20M17, 06F05.
Bundit Pibaljommee is the corresponding author.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

role in this direction. One of the generalizations of one-sided ideals of ordered semigroups is the notion of bi-ideals, introduced by Kehayopulu in [4]. Many regularities of ordered semigroups were characterized using their ideals by some authors [7, 9, 10]. One idea of ideals in ordered semigroups that is incomparable with bi-ideals is the notion of interior ideals. Kehayopulu started this concept in [5]. It was illustrated that, in general, a bi-ideal need not be an interior ideal, and vice versa. Moreover, the author proved that the concept of ideals and interior ideals coincide in some particular classes of ordered semigroups. Bussaban and Changphas [1] extended the notion of bi-ideals of ordered semigroups to the notion of (m,n)-ideals and provided a characterization of (m,n)-ideals of an (m,n)-ideal being an (m,n)-ideal of an ordered semigroup.

In this paper, we try something similar to the work of Bussaban and Changphas [1] by introducing the concept of n-interior ideals of ordered semigroups. It turns out that this notion is a generalization of interior ideals introduced by Kehayopulu [5]. By the incomparable of bi-ideals and interior ideals of ordered semigroup, it is evident that the notion of (m,n)-ideals and n-ideals of ordered semigroups do not become relevant in general. For this reason, the central theme of our paper is to combine these two notions to characterize many regularities of ordered semigroups.

2 Preliminaries

An *ordered semigroup* is an algebraic structure $(S,\cdot,\leq)$ such that $(S,\cdot)$ is a semigroup, and $(S,\leq)$ is a partially ordered set, where $\leq$ is compatible with the operation $\cdot$. That is, for any $x,y\in S$ with $x\leq y$, we have that $x\cdot z\leq y\cdot z$ and $z\cdot x\leq z\cdot y$, for all $z\in S$. For simplicity, we write the product $x\cdot y$ of x and y as xy. In this paper, we denote an ordered semigroup $(S,\cdot,\leq)$ by S of its underlying set unless specified otherwise. Let A,B be subsets of S and let C be a nonempty subset of S. Define the sets AB and $(C]$ as follows: $AB:=\emptyset$ if A or B is empty, otherwise, $AB:=\{ab : a\in A \text{ and } b\in B\}$ and $(C]:=\{x\in S : x\leq c \text{ for some } c\in C\}$. One can easily prove that $A(B\cup C)=AB\cup AC$, for all $A,B,C\subseteq S$.

Lemma 2.1. *([3]) Let A and B be nonempty subsets of S. Then the following statements are valid.*

(1) $A\subseteq (A]$ *and* $((A]]=(A]$. (2) $A\subseteq B$ *implies* $(A]\subseteq (B]$.
(3) $A(B],(A]B\subseteq (A](B]\subseteq (AB]$. (4) $(A\cup B]=(A]\cup(B]$.

By a *subsemigroup* of S, we mean a nonempty subset A of S such that $AA \subseteq A$. A subsemigroup A of S such that $A = (A]$ is called an *interior ideal* of S if $SAS \subseteq A$, and a *bi-ideal* of S if $ASA \subseteq A$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m, n \in \mathbb{N}_0$. A subsemigroup A of S is called an (m, n)-*ideal* [1] of S if $A^m SA^n \subseteq A$ and $A = (A]$.

Here, $A^0 S = A^0 SA^0 = SA^0 = S$. We observe that $(1, 1)$-ideal is a bi-ideal of S.

Let $\emptyset \neq A \subseteq S$, we denote by $I(A)$, $B(A)$ and $I_{(m,n)}(A)$ the interior ideal, the bi-ideal and the (m, n)-ideal of S generated by A, respectively. One can show that $I(A) = (A \cup A^2 \cup SAS]$, $B(A) = (A \cup A^2 \cup ASA]$ and $I_{(m,n)}(A) = (A \cup \cdots \cup A^{m+n} \cup A^m SA^n]$ (see [1], [2], [6]). We use $I(a), B(a)$ and $I_{(m,n)}(a)$ instead of $I(\{a\}), B(\{a\})$ and $I_{(m,n)}(\{a\})$, respectively.

Phochai and Changphas [8] divided semigroups into sixteen types of regularities as follows:
An ordered semigroup S is said to be

(C1) for all $a \in S$, we have $a \in (SaS]$, (*intra-reproduce*)
(equivalently, $A \subseteq (SAS]$ for any $\emptyset \neq A \subseteq S$),

(C2) for all $a \in S$, we have $a \in (Sa]$, (*left reproduce*)
(equivalently, $A \subseteq (SA]$ for any $\emptyset \neq A \subseteq S$),

(C3) for all $a \in S$, we have $a \in (aS]$, (*right reproduce*)
(equivalently, $A \subseteq (AS]$ for any $\emptyset \neq A \subseteq S$),

(C4) for all $a \in S$, we have $a \in (SaSaS]$, (*intra-quasi-regular*)
(equivalently, $A \subseteq (SASAS]$ for any $\emptyset \neq A \subseteq S$),

(C5) for all $a \in S$, we have $a \in (SaSa]$, (*left quasi-regular*)
(equivalently, $A \subseteq (SASA]$ for any $\emptyset \neq A \subseteq S$),

(C6) for all $a \in S$, we have $a \in (aSaS]$, (*right quasi-regular*)
(equivalently, $A \subseteq (ASAS]$ for any $\emptyset \neq A \subseteq S$),

(C7) for all $a \in S$, we have $a \in (aSa]$, (*regular*)
(equivalently, $A \subseteq (ASA]$ for any $\emptyset \neq A \subseteq S$),

(C8) for all $a \in S$, we have $a \in (Sa^k S]$ for some $k \in \mathbb{N} \setminus \{1\}$, (equivalently, $A \subseteq (SA^k S]$ for any $\emptyset \neq A \subseteq S$ and for some $k \in \mathbb{N} \setminus \{1\}$),

(C9) for all $a \in S$, we have $a \in (Sa^k Sa]$ for some $k \in \mathbb{N} \setminus \{1\}$, (equivalently, $A \subseteq (SA^k SA]$ for any $\emptyset \neq A \subseteq S$ and for some $k \in \mathbb{N} \setminus \{1\}$), (C10) for all $a \in S$, we have $a \in (aSa^kS]$ for some $k \in \mathbb{N} \setminus \{1\}$, (equivalently, $A \subseteq (ASA^kS]$ for any $\emptyset \neq A \subseteq S$ and for some $k \in \mathbb{N} \setminus \{1\}$),

(C11) for all $a \in S$, we have $a \in (aSa^kSa]$ for some $k \in \mathbb{N}$, (equivalently, $A \subseteq (ASA^kSA]$ for any $\emptyset \neq A \subseteq S$ and for some $k \in \mathbb{N}$),

(C12) for all $a \in S$, we have $a \in (Sa^2]$, (*left regular*) (equivalently, $A \subseteq (SA^2]$ for any $\emptyset \neq A \subseteq S$),

(C13) for all $a \in S$, we have $a \in (a^2S]$, (*right regular*) (equivalently, $A \subseteq (A^2S]$ for any $\emptyset \neq A \subseteq S$),

(C14) for all $a \in S$, we have $a \in (a^2Sa^2]$, (*completely regular*) (equivalently, $A \subseteq (A^2SA^2]$ for any $\emptyset \neq A \subseteq S$),

(C15) for all $a \in S$, we have $a \in (aSa^2]$, (equivalently, $A \subseteq (ASA^2]$ for any $\emptyset \neq A \subseteq S$),

(C16) for all $a \in S$, we have $a \in (a^2Sa]$, (equivalently, $A \subseteq (A^2SA]$ for any $\emptyset \neq A \subseteq S$).

Note that S satisfying $(C8)$ with $k = 2$ is called *intra-regular*.

3 Characterization ordered semigroups

In this section, we define the concept of an n-interior ideal of ordered semigroups. Then we characterize thirteen classes of ordered semigroups, classified by linear inequalities in terms of (m, n)-ideals and n-interior ideals.

Now, we define the notion of an n-interior ideal of an ordered semigroup.

Definition 3.1. *Let $n \in \mathbb{N}_0$. A subsemigroup A of S is called an* n-interior ideal *of S if $A = (A]$ and $SA^nS \subseteq A$.*

Since $SA^0 = S$ and $SA^0S = S^2$, A is a 0-interior ideal of S if and only if $S^2 \subseteq A$. We note that 1-interior ideal is an interior ideal of S.

Example 3.2. *Let $S = \{a, b, c, d, e, f\}$. Define a binary operation $\cdot$ on S and a partial order $\leq$ on S as follows:*

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	a	b
c	a	a	a	a	a	b
d	a	a	a	a	a	d
e	a	d	d	a	a	d
f	a	d	d	d	e	f

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,d)\}.$$

Then $(S,\cdot,\leq)$ is an ordered semigroup. Now, we have $\{a,b\}$ is an n-interior ideal, where $n \geq 2$ but not a 1-interior ideal.

For any nonempty subset A of S, we denote by $I_n(A)$ the n-interior ideal of S generated by A. If $A = \{a\}$, then we write $I_n(a)$ instead of $I_n(\{a\})$.

Theorem 3.3. *Let $\emptyset \neq A \subseteq S$ and $n \in \mathbb{N}_0$. Then $I_n(A) := (A \cup \cdots \cup A^{n+1} \cup SA^nS]$ is the n-interior ideal of S generated by A.*

Proof. By Lemma 2.1, $(A \cup \cdots \cup A^{n+1} \cup SA^nS](A \cup \cdots \cup A^{n+1} \cup SA^nS] \subseteq ((A \cup \cdots \cup A^{n+1} \cup SA^nS)(A \cup \cdots \cup A^{n+1} \cup SA^nS)] = (A^2 \cup \cdots \cup A^{n+2} \cup ASA^nS) \cup (A^3 \cup \cdots \cup A^{n+3} \cup A^2SA^nS) \cup \cdots \cup (A^{n+2} \cup \cdots \cup A^{2n+2} \cup A^{n+1}SA^nS) \cup (SA^nSA \cup \cdots \cup SA^nSSA^nS] \subseteq (A \cup \cdots \cup A^{n+1} \cup SA^nS]$. Therefore, $(A \cup \cdots \cup A^{n+1} \cup SA^nS]$ is a subsemigroup of S. By Lemma 2.1, we have $S(A \cup \cdots \cup A^{n+1} \cup SA^nS]^nS \subseteq (SA^nS] \subseteq (A \cup \cdots \cup A^{n+1} \cup SA^nS]$. Hence, $(A \cup \cdots \cup A^{n+1} \cup SA^nS]$ is an n-interior ideal containing A. Finally, If B is an n-interior ideal of S containing A, then $(A \cup \cdots \cup A^{n+1} \cup SA^nS] \subseteq (B \cup \cdots \cup B^{n+1} \cup SB^nS] \subseteq B$. □

Theorem 3.4. *S is intra-reproduced if and only if $S \cap I \subseteq (SIS]$ for all 1-interior ideal I of S.*

Proof. Assume that S is intra-reproduced. Let I be a 1-interior ideal of S. Then $S \cap I \subseteq (S(S \cap I)S] \subseteq (SIS]$. Conversely, let $a \in S$. Then $a \in S \cap I(a) \subseteq (SI(a)S] = (S(a \cup a^2 \cup SaS]S] \subseteq (SaS]$. Therefore, S is intra-reproduced. □

Theorem 3.5. *S is intra-quasi-regular if and only if $I \cap B \subseteq (IBI]$ for any 1-interior ideal I and $(1,1)$-ideal B of S.*

Proof. Assume that S is intra-quasi-regular. Let I be a 1-interior ideal and let B be a $(1,1)$-ideal of S. By Lemma 2.1, $I \cap B \subseteq (S(I \cap B)S(I \cap B)S] \subseteq (SIS(I \cap B)S] \subseteq (SIS(S(I \cap B)S(I \cap B)S]S] \subseteq (SISSBSISS] \subseteq (IBI]$.

Conversely, let $a \in S$. Then $a \in I(a) \cap B(a) \subseteq (I(a)B(a)I(a)] = ((a \cup a^2 \cup SaS](a \cup a^2 \cup aSa](a \cup a^2 \cup SaS]] \subseteq (a^3 \cup a^4 \cup a^2SaS \cup Sa^2 \cup Sa^3 \cup SaSaS] \subseteq (Sa^2 \cup SaSaS] = (Sa^2] \cup (SaSaS]$. If $a \in (Sa^2]$, then there exists $x \in S$, $a \leq xaa \leq xxaaxaa$, i.e., $a \in (SaSaS]$. Therefore, S is intra-quasi-regular. □

Theorem 3.6. *S is left-quasi-regular if and only if $I \cap B \subseteq (IB]$ for any 1-interior ideal I and $(1, 1)$-ideal B of S.*

Proof. Assume that S is intra-quasi-regular. Let I be a 1-interior ideal and B be a $(1, 1)$-ideal of S. Then $I \cap B \subseteq (S(I \cap B)S(I \cap B)] \subseteq (SISB] \subseteq (IB]$. Conversely, let $a \in S$. Then $a \in I(a) \cap B(a) \subseteq (I(a)B(a)] = ((a \cup a^2 \cup SaS](a \cup a^2 \cup aSa]] \subseteq (a^2 \cup aSa \cup Sa^2 \cup SaSa] = (a^2] \cup (aSa] \cup (Sa^2] \cup (SaSa]$. If $a \in (a^2]$, then $a \leq aa \leq aaaa$, i.e., $a \in (SaSa]$. If $a \in (aSa]$, then there exists $x \in S$, $a \leq axa \leq axaxaxa$, i.e., $a \in (SaSa]$. If $a \in (Sa^2]$, then there exists $x \in S$, $a \leq xaa \leq xxaaxaa$; i.e., $a \in (SaSa]$. Therefore, S is left-quasi-regular. □

Corollary 3.7. *S is right-quasi-regular if and only if $I \cap B \subseteq (BI]$ for any 1-interior ideal I and $(1, 1)$-ideal B of S.*

Theorem 3.8. *S is regular if and only if $I \cap B \subseteq (BIB]$ for any 1-interior ideal I and $(1, 1)$-ideal B of S.*

Proof. Assume that S is regular. Let I be a 1-interior ideal and let B be a $(1, 1)$-ideal of S. Then $I \cap B \subseteq ((I \cap B)S(I \cap B)] \subseteq ((I \cap B)S(I \cap B)S(I \cap B)S(I \cap B)] \subseteq (BSISBSB] = (BIB]$. Conversely, let $a \in S$. Then $a \in I(a) \cap B(a) \subseteq (B(a)I(a)B(a)] = ((a \cup a^2 \cup aSa](a \cup a^2 \cup SaS](a \cup a^2 \cup aSa]] \subseteq (aSa]$. Therefore, S is regular. □

Remark 3.9. *Let $a \in S$ and $1 \neq m \in \mathbb{N}$. If $a \leq a^m$, then $a \leq a^{km-k+1}$ for all $k \in \mathbb{N}$.*

Theorem 3.10. *S is $C8$ if and only if $A \subseteq (A^2]$ for any k-interior ideal A of S.*

Proof. Assume that S is $C8$. Let A be a k-interior ideal of S. By Lemma 2.1, $A \subseteq (SA^kS] \subseteq (S(SA^kS](SA^kS]S] \subseteq ((SA^kS](SA^kS]] \subseteq (AA]$. Conversely, let $a \in S$. Then $a \in I_k(a) \subseteq (I_k(a)I_k(a)] = ((a \cup a^2 \cup \cdots \cup a^{k+1} \cup Sa^kS](a \cup a^2 \cup \cdots \cup a^{k+1} \cup Sa^kS]] \subseteq (a^2 \cup a^3 \cup \cdots \cup a^k \cup a^{k+1} \cup Sa^kS] = (a^2] \cup (a^3] \cup \cdots \cup (a^k] \cup (a^{k+1}] \cup (Sa^kS]$. By Remark 3.9, $a \in (Sa^kS]$. Therefore, S is $C8$. □

Theorem 3.11. *S is $C9$ if and only if $A \cap E \subseteq (AE]$ for any k-interior ideal A and $(k,1)$-ideal E of S.*

Proof. Assume that S is $C9$. Let A be a k-interior ideal and E a $(k,1)$-ideal of S. Then $A \cap E \subseteq (S(A\cap E)^k S(A\cap E)] \subseteq ((SA^kS)E] \subseteq (AE]$. Conversely, let $a \in S$. Then $a \in I_k(a) \subseteq (I_k(a)I_{(k,1)}(a)] = ((a \cup a^2 \cup \cdots \cup a^{k+1} \cup Sa^kS](a \cup a^2 \cup \cdots \cup a^{k+1} \cup a^kSa]] \subseteq (a^2 \cup a^3 \cup \cdots \cup a^k \cup a^{k+1} \cup a^{k+2} \cup Sa^kSa] = (a^2] \cup (a^3] \cup \cdots \cup (a^k] \cup (a^{k+1}] \cup (a^{k+2}] \cup (Sa^kSa]$. By Remark 3.9, $a \in (Sa^kSa]$. Therefore, S is $C9$. □

Corollary 3.12. *S is $C10$ if and only if $A \cap F \subseteq (FA]$ for any k-interior ideal A and $(1,k)$-ideal F of S.*

Theorem 3.13. *S is left regular if and only if $I \cap B \cap C \subseteq (IBC]$ for any 1-interior ideal I, $(1,1)$-ideal B and $(1,2)$-ideal C of S.*

Proof. Assume that S is left regular. Let I be a 1-interior ideal, B a $(1,1)$-ideal and C a $(1,2)$-ideal of S. By Lemma 2.1, $I \cap B \cap C \subseteq (S(I\cap B\cap C)^2] \subseteq (S(I \cap B \cap C)(S(I \cap B \cap C)^2] \subseteq (SISBC] \subseteq (IBC]$. Conversely, let $a \in S$. Then $a \in I(a) \cap B(a) \cap I_{(1,2)}(a) \subseteq (I(a)B(a)I_{(1,2)}(a)] = ((a \cup a^2 \cup SaS](a \cup a^2 \cup aSa](a \cup a^2 \cup a^3 \cup aSa^2]] \subseteq (Sa^2 \cup Sa^3 \cup Sa^4 \cup Sa^2Sa^2] \subseteq (Sa^2]$. Therefore, S left regular. □

Corollary 3.14. *S is right regular if and only if $I \cap B \cap D \subseteq (DBI]$ for any 1-interior ideal I, $(1,1)$-ideal B and $(2,1)$-ideal D of S.*

Theorem 3.15. *S is completely regular if and only if $D \cap C \subseteq (DC]$ for any $(2,1)$-ideal D and $(1,2)$-ideal C of S.*

Proof. Assume that S is completely regular. Let D be a $(2,1)$-ideal and let C be a $(1,2)$-ideal of S. Then $D \cap C \subseteq ((D\cap C)^2 S(D \cap C)^2] \subseteq (DCSCC] \subseteq (DC]$.
Conversely, let $a \in S$. Then $a \in I_{(2,1)}(a) \cap I_{(1,2)}(a) \subseteq (I_{(2,1)}(a)I_{(1,2)}(a)] = ((a \cup a^2 \cup a^3 \cup a^2Sa](a \cup a^2 \cup a^3 \cup aSa^2]] \subseteq (a^2 \cup a^3 \cup a^4 \cup a^2Sa^2] = (a^2] \cup (a^3] \cup (a^4] \cup (a^2Sa^2]$. By Remark 3.9, $a \in (a^2Sa^2]$. So, S is completely regular. □

Theorem 3.16. *S is $C15$ if and only if $C \subseteq (C^2]$, for any $(1,2)$-ideal C of S.*

Proof. Assume that S is $C15$. Let C be a $(1,2)$-ideal of S. By Lemma 2.1, $C \subseteq (CSC^2] \subseteq (CS(CSCC]C] \subseteq (CSCSC^3] \subseteq (CC]$. Conversely, let $a \in S$. Then $a \in I_{(1,2)}(a) \subseteq (I_{(1,2)}(a)I_{(1,2)}(a)] = ((a \cup a^2 \cup a^3 \cup aSa^2](a \cup a^2 \cup a^3 \cup aSa^2]] \subseteq (a^2 \cup a^3 \cup aSa^2] = (a^2] \cup (a^3] \cup (aSa^2]$. By Remark 3.9, $a \in (aSa^2]$. Therefore, S is $C15$. □

Corollary 3.17. *S is $C16$ if and only if $D \subseteq (D^2]$ for any $(2,1)$-ideal D of S.*

References

[1] L. Bussaban, T. Changphas, On (m,n)-ideals and (m,n)-regular ordered semigroups, Songklanakarin J. Sci. Technol., **38,** no. 2, (2016), 199–206.

[2] T. Changphas, On classes of regularity in an ordered semigroup, Quasigroups Related Systems, **21,** (2013), 43–48.

[3] N. Kehayopulu, On weakly prime ideals of ordered semigroups, Math. Japon., **35,** no. 6, (1990), 1051–1056.

[4] N. Kehayopulu, On completely regular *poe*-semigroups, Math. Japon., **37,** no. 1, (1992), 123–130.

[5] N. Kehayopulu, Interior ideals and interior ideal elements in ordered semigroups, Pure Math. Appl., **2,** no. 3, (1999), 407–409.

[6] A. Khan, Y. B. Jun, M. Z. Abbas, Characterizations of ordered semigroups in terms of $(\epsilon, \epsilon \vee q)$-fuzzy interior ideals, Neural comput. Appl., **21,** (2012), 433–440.

[7] D. M. Lee, S. K. Lee, On intra-regular ordered semigroups, Kangweon-Kyungki Math. J., **14,** no. 1, (2006), 95–100.

[8] T. Phochai, T. Changphas, Linear inequations and regularity conditions on ordered semigroups, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), LXII(f.3), (2017), 627–636.

[9] M. Shabir, A. Khan, Characterizations of ordered semigroups by the properties of their fuzzy ideals, Comput. Math. Appl., (2010), 539–549.

[10] X. Y. Xie, J. Tang, Regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy subsets, Iran. J. Fuzzy Syst., **7,** no. 2, (2010), 121–140.