International Journal of Mathematics and
Computer Science, **17**(2022), no. 2, 717–722

Pasting Lemmas for b-Metric Preserving and Related Functions II

Tammatada Khemaratchatakumthorn, Duangpon Siriwan

Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom, 73000, Thailand

email: tammatada@gmail.com, khemaratchataku_t@silpakorn.edu
duangpon.siriwan@gmail.com,

(Received October 9, 2021, Accepted November 16, 2021)

Abstract

We continue the investigation from our previous article on the pasting lemmas for b-metric preserving and related functions.

1 Introduction

A well-known and simple version of the pasting lemma states that if $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are continuous and $g(b) = h(b)$, then the function $f : [a, c] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x), & \text{if } x \in [a, b); \\ h(x), & \text{if } x \in [b, c] \end{cases}$$

is also continuous. A pasting lemma for metric-preserving functions was given by Doboš [3, p. 26] and those for b-Metric-preserving and some related functions were given by us in a previous article [6]. In this paper, we continue the investigation and give pasting lemmas for the remaining functions in Figure 1.

Key words and phrases: Metric, b-metric, ultrametric, metric-preserving function, pasting lemma.
AMS (MOS) Subject Classifications: 26A21, 26A30.
Tammatada Khemaratchatakumthorn is the corresponding author.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

Recall that $\mathcal{M}$ is the set of all metric-preserving functions, $\mathcal{U}$ is the set of all ultrametric-preserving functions, $\mathcal{UM}$ is the set of all ultrametric-metric-preserving functions, $\mathcal{MU}$ is the set of all metric-ultrametric-preserving functions, $\mathcal{B}$ is the set of all b-Metric-preserving functions, $\mathcal{MB}$ is the set of all metric-b-Metric-preserving functions, $\mathcal{BM}$ is the set of all b-Metric-metric-preserving functions, $\mathcal{UB}$ is the set of all ultrametric-b-Metric-preserving functions, and $\mathcal{BU}$ is the set of all b-Metric-ultrametric-preserving functions. For more information, we refer the reader to [2] for various generalizations of metrics, to [1, 3, 7] for metric-preserving functions, to [8] for the functions in $\mathcal{U}$, $\mathcal{UM}$, and $\mathcal{MU}$, to [4, 5, 9] for the functions in $\mathcal{B}$, $\mathcal{MB}$, $\mathcal{BM}$, $\mathcal{UB}$, and $\mathcal{BU}$. In particular, the authors of [9] obtained all subset relations between these classes of functions as follows:

Theorem 1.1. [9] *The following statements hold.*

(i) $\mathcal{MU} = \mathcal{BU} \subseteq \mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} = \mathcal{MB} \subseteq \mathcal{UB}$.

(ii) $\mathcal{BU} = \mathcal{MU} \subseteq \mathcal{U} \subseteq \mathcal{UM} \subseteq \mathcal{UB}$.

(iii) $\mathcal{M} \subseteq \mathcal{UM}$.

The subset relations in Theorem 1.1 can be summarized in the diagram (Figure 1). Note that $f \in A \Rightarrow f \in B$ means $f \in A$ implies $f \in B$. In addition, if there is no arrow from $f \in A$ to $f \in B$, it means that $A \nsubseteq B$.

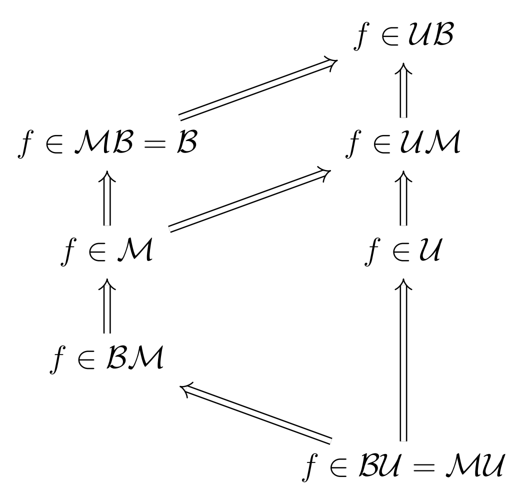

Figure 1: Subset Relations

In the previous article, we [6] give pasting lemmas for $\mathcal{B}$, $\mathcal{MB}$, and $\mathcal{BM}$, which are the classes of functions on the left of the diagram (Figure 1). Hence, to complete the investigation, we give pasting lemmas for the remaining classes of functions on the right of the diagram.

2 Preliminaries and Lemmas

Recall that a function $f : [0, \infty) \to [0, \infty)$ is said to be amenable if $f^{-1}(\{0\}) = \{0\}$. In order to prove our main theorem, we need some useful theorems in [8] and [9] as follows.

Lemma 2.1. [8] *If $f \in \mathcal{MU}$ if and only if f is amenable and constant on $(0, \infty)$.*

Lemma 2.2. [8] *If $f \in \mathcal{U}$ if and only if f is amenable and increasing on $[0, \infty)$.*

Lemma 2.3. [8] *Let f be amenable. Then $f \in \mathcal{UM}$ if and only if $f(a) \leq 2f(b)$ for all $0 \leq a \leq b$.*

Lemma 2.4. [9] *If $f \in \mathcal{UB}$, then f is amenable. In additional, if f is amenable, then $f \in \mathcal{UB}$ if and only if there exists $s \geq 1$ such that $f(a) \leq sf(b)$ whenever $0 \leq a \leq b$.*

3 Main Results

In this section, we give pasting lemmas for the functions in $\mathcal{MU}$, $\mathcal{BU}$, $\mathcal{U}$, $\mathcal{UM}$, and $\mathcal{UB}$.

Theorem 3.1. (A pasting lemma for functions in $\mathcal{MU}$ and $\mathcal{BU}$) *Let $g, h : [0, \infty) \to [0, \infty)$, g, $h \in \mathcal{MU}$, $r > 0$ and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Then $f \in \mathcal{MU}$.

Proof. By Lemma 2.1, g and h are amenable and constant on $(0, \infty)$. Since $g(r) = h(r)$, we have $g(x) = g(r) = h(r) = h(x)$ for all $x > 0$. So f is amenable and constant on $(0, \infty)$. Therefore $f \in \mathcal{MU}$. □

Theorem 3.2. (A pasting lemma for functions in $\mathcal{U}$) *Let* $g, h : [0,\infty) \to [0,\infty)$, g, $h \in \mathcal{U}$, $r > 0$ *and* $g(r) = h(r)$. *Define* $f : [0,\infty) \to [0,\infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,r), \\ h(x), & \text{if } x \in [r,\infty). \end{cases}$$

Then $f \in \mathcal{U}$.

Proof. By Lemma 2.2, g and h are amenable and increasing on $[0,\infty)$. Since $g(r) = h(r)$ and h is increasing, it follows that f is increasing. Since g is amenable, so is f. Therefore $f \in \mathcal{U}$. □

Theorem 3.3. (A pasting lemma for functions in $\mathcal{UM}$) *Let* $g, h : [0,\infty) \to [0,\infty)$, $g, h \in \mathcal{UM}$, $r > 0$ *and* $g(r) = h(r)$. *Define* $f : [0,\infty) \to [0,\infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,r), \\ h(x), & \text{if } x \in [r,\infty). \end{cases}$$

Then $f \in \mathcal{UM}$ *if and only if* $\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x)$.

Proof. We use Lemma 2.3 throughout the proof without further reference. Assume $f \in \mathcal{UM}$. Since $g(x) \le 2g(r)$ for every $x \in (0,r)$, we see that $\sup_{x\in(0,r)} g(x)$ exists. Similarly, $h(x) \ge \frac{1}{2}h(r)$ for every $x \in [r,\infty)$, so we obtain that $\inf_{x\in[r,\infty)} h(x)$ exists. Let $x \in (0,r)$ and $y \in [r,\infty)$. Then $x \le y$ and

$$g(x) = f(x) \le 2f(y) = 2h(y).$$

Since $g(x) \le 2h(y)$ for all $x \in (0,r)$, we obtain $\sup_{x\in(0,r)} g(x) \le 2h(y)$. Since $\sup_{x\in(0,r)} g(x) \le 2h(y)$ for all $y \in [r,\infty)$, we have

$$\sup_{x\in(0,r)} g(x) \le \inf_{y\in[r,\infty)} 2h(y) = 2\inf_{y\in[r,\infty)} h(y) = 2\inf_{x\in[r,\infty)} h(x).$$

For the converse, assume that $\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x)$. Let $0 \le a \le b$. If $a, b < r$, then $f(a) = g(a) \le 2g(b) = 2f(b)$. If $a, b \ge r$, then $f(a) = h(a) \le 2h(b) = 2f(b)$. So suppose that $a < r \le b$. Then

$$f(a) = g(a) \le \sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x) \le 2h(b) = 2f(b).$$

In any case, $f(a) \le 2f(b)$. Hence $f \in \mathcal{UM}$. This completes the proof. □

Theorem 3.4. (A pasting lemma for functions in $\mathcal{UB}$) *Let* $g, h : [0,\infty) \to [0,\infty)$*,* $g, h \in \mathcal{UB}$*,* $r > 0$ *and* $g(r) = h(r)$*. Define* $f : [0,\infty) \to [0,\infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,r), \\ h(x), & \text{if } x \in [r,\infty). \end{cases}$$

Then $f \in \mathcal{UB}$*.*

Proof. Since g and h are amenable, it is easy to see that f is amenable. By Lemma 2.4, there are $s_1, s_2 \geq 1$ such that $g(a) \leq s_1 g(b)$ and $h(a) \leq s_2 h(b)$ for all $0 \leq a \leq b$. Since $g(x) \leq s_1 g(r)$ for every $x \in (0,r)$, we see that $\sup_{x\in(0,r)} g(x)$ exists. Similarly, $h(x) \geq \frac{1}{s_2} h(r)$ for every $x \in [r,\infty)$, so $\inf_{x\in[r,\infty)} h(x)$ exists and is positive. Then there exists $s_3 \geq 1$ such that

$$\sup_{x\in(0,r)} g(x) \leq s_3 \inf_{x\in[r,\infty)} h(x).$$

To show that $f \in \mathcal{UB}$, we choose $s = \max\{s_1, s_2, s_3\}$ and let $0 \leq a \leq b$. If $a, b < r$, then $f(a) = g(a) \leq s_1 g(b) \leq s g(b) = s f(b)$. If $a, b \geq r$, then $f(a) = h(a) \leq s_2 h(b) \leq s h(b) = s f(b)$. So suppose that $a < r \leq b$. Then

$$f(a) = g(a) \leq \sup_{x\in(0,r)} g(x) \leq s_3 \inf_{x\in[r,\infty)} h(x) \leq s \inf_{x\in[r,\infty)} h(x) \leq s h(b) = s f(b).$$

In any case, we have $f(a) \leq s f(b)$. Therefore $f \in \mathcal{UB}$, as desired and so the proof is complete. □

Acknowledgments. We would like to thank Prapanpong Pongsriiam for his support. Tammatada Khemaratchatakumthorn received financial support from Faculty of Science, Silpakorn University, Thailand, grant number SRIF-JRG-2563-03.

References

[1] P. Corazza, *Introduction to metric-preserving functions*, American Mathematical Monthly, **106,** no. 4, (1999), 309–323.

[2] M. M. Deza, E. Deza, *Encyclopedia of Distances*, Second edition, Springer, 2013.

[3] J. Doboš, *Metric Preserving Functions*, Online Lecture Notes available at http://web.science.upjs.sk/jozefdobos/wp-content/uploads/2012/03/mpf1.pdf

[4] T. Khemaratchatakumthorn, P. Pongsriiam, *Remarks on b-Metric and metric-preserving functions*, Mathematica Slovaca, **68,** no. 5, (2018), 1009–1016.

[5] T. Khemaratchatakumthorn, P. Pongsriiam, S. Samphavat, *Further remarks on b-Metrics, metric-preserving functions, and other related metrics*, International Journal of Mathematics and Computer Science, **14,** no. 2, (2019), 473–480.

[6] T. Khemaratchatakumthorn, D. Siriwan, *Pasting lemmas for bMetric preserving and related functions*, International Journal of Mathematics and Computer Science, **16,** no. 4, (2021), 1591–1598.

[7] P. Pongsriiam, I. Termwuttipong, *On metric-preserving functions and fixed point theorems*, Fixed Point Theory and Applications, 2014:179, 14 pages

[8] P. Pongsriiam, I. Termwuttipong, *Remarks on ultrametrics and metric-preserving functions*, Abstract and Applied Analysis Article ID 163258, 2014, 9 pages.

[9] S. Samphavat, T. Khemaratchatakumthorn, P. Pongsriiam, *Remarks on b-Metrics, ultrametrics, and metric-preserving functions*, Mathematica Slovaca, **70,** no. 1, 2020, 61–70.