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Subsequences and Exact Divisibility by the Powers of the Balancing Numbers

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Abstract

We study certain subsequences of the balancing numbers and determine their residues modulo the powers of the balancing numbers. This leads to an exact divisibility theorem for the subsequence and also gives an analogous result to those of the Fibonacci numbers.

1 Introduction

A positive integer n is said to be a balancing number if $n = 0$, $n = 1$, or n satisfies the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

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for some $r \in \mathbb{N}$. For example, $n = 6$ is a balancing number since $1+2+3+4+5 = 7+8$. Then the sequence $(B_n)_{n\geq 0}$ of the balancing numbers can be obtained from the recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ when $n \geq 2$ with $B_0 = 0$ and $B_1 = 1$. Recall also that, for integers $k \geq 0$ and $m, n \geq 2$, we say that m^k exactly divides n and write $m^k \parallel n$ if $m^k \mid n$ and $m^{k+1} \nmid n$.

In this article, we are interested in studying exact divisibility and the residues modulo the powers of the balancing numbers in the sequence $(G(k, n, m))_{k\geq 1}$ defined as follows:
Let $m \geq 1$ and $n \geq 2$ be integers. Define

$$G(1, n, m) = B_n^m \text{ and } G(k, n, m) = B_{nG(k-1,n,m)} \quad \text{for } k \geq 2.$$

Therefore, $(G(k, n, m))_{k\geq 2}$ is a subsequence of the balancing numbers and we would like to determine the residue of $G(k, n, m)/B_n^\ell$ modulo B_n, where ℓ is the largest integer such that B_n^ℓ divides $G(k, n, m)$.

Our motivation is to obtain analogous results to those of Fibonacci numbers as first appeared in Matijasevich's solution to Hilbert's 10th problem [6, 7, 8] which were further investigated by Panraksa, Tangboonduangjit, and Wiboonton [15] and extended by Onphaeng and Pongsriiam [9, 10, 11, 12] and Pongsriiam [17]. Notice that Panraksa and Tangboonduangjit [14] have recently obtained the exact divisibility result for a subsequence of the Lucas sequence of the first kind which covers our corollary. However, our main theorem is new and gives an idea on how to extend it to the complete calculation of the residues in the Lucas sequence of the first kind.

For more information on the balancing numbers, see for example the work by Behera and Panda [2], Panda [13], Patra, Panda, and Khemaratchatakumthorn [16], Ray [21], and Sequence A001109 in OEIS [23]. For some recent results on the divisibility properties of Fibonacci numbers and generalizations, we refer the reader to the articles by Ballot [1], Cubre and Rouse [3], Khaochim and Pongsriiam [4, 5], Pongsriiam [18, 19, 20], Sanna [22], and Stewart [24].

2 Preliminaries and lemmas

Unless stated otherwise, k, m, n are integers and $m, n \geq 1$. We first recall that if p is a prime, we write $\nu_p(n)$ to denote the p-adic valuation of n which is defined to be the nonnegative integer k such that $p^k \parallel n$. Next, we give some useful results which are needed in the proof of our main theorem.

Lemma 2.1. (Panda [13], Ray [21]) *The following statements hold:*

(i) $B_n \mid B_m$ *if and only if* $n \mid m$. *Moreover,* $\gcd(B_m, B_n) = B_{\gcd(m,n)}$.

(ii) $B_{n-1}B_{n+1} - B_n^2 = -1$.

Lemma 2.2. (Patra, Panda, and Khemaratchatakumthorn [16]) *We have*

$$B_{mn} = \sum_{j=1}^{n} \binom{n}{j} B_m^j B_{m-1}^{n-j}(-1)^{n-j}B_j = \sum_{j=1}^{m} \binom{m}{j} B_n^j B_{n-1}^{m-j}(-1)^{m-j}B_j.$$

Lemma 2.3. (Onphaeng and Pongsriiam [11]) *Let* k, ℓ, m, s *be positive integers and* $s^k \mid m$. *Then* $s^{k+\ell} \mid \binom{m}{j}s^j$, *for all* $1 \le j \le m$ *satisfying* $2^{j-\ell+1} > j$. *In particular,* $s^{k+1} \mid \binom{m}{j}s^j$, *for all* $1 \le j \le m$, *and* $s^{k+2} \mid \binom{m}{j}s^j$, *for all* $3 \le j \le m$.

Lemma 2.4. (Patra, Panda, and Khemaratchatakumthorn [16]) *For all* $k \ge 1$ *and* $m, n \ge 2$, *we have* $B_n^k \mid m$ *if and only if* $B_n^{k+1} \mid B_{nm}$.

Lemma 2.5. (Patra, Panda, and Khemaratchatakumthorn [16]) $v_2(B_n) = v_2(n)$.

Lemma 2.6. *Let* n *be a positive integer. Then* $B_{n+1}^2 \equiv B_{n-1}^2 \equiv 1 \pmod{B_n}$.

Proof. Replacing n by $n-1$ in Lemma 2.1(ii) and reducing the equation modulo B_n, we get the desired result. □

Lemma 2.7. *Let* k, m, n *be positive integers. Then* $G(k, n, m)$ *is even if and only if* n *is even.*

Proof. We apply Lemma 2.5 throughout the proof. Assume that n is even. Then B_n is even and so $G(1, n, m)$ is even. If $k \ge 2$, then

$$v_2(G(k, n, m)) = v_2(B_{nG(k-1,n,m)}) = v_2(nG(k-1, n, m)) \ge 1.$$

This implies $G(k, n, m)$ is even for every $k \ge 1$. Next, suppose n is odd. Then $G(1, n, m) = B_n^m$ is odd and if $k \ge 1$ and $G(k, n, m)$ is odd, then $G(k+1, n, m) = B_{nG(k,n,m)}$ is odd. Therefore, the result follows by induction on k. □

3 Main results

We are now ready to prove our main theorem.

Theorem 3.1. *Let m and n be positive integers. Then the following statements hold:*

(i) $B_n^{m+k-1} \mid G(k,n,m)$, *for every* $k \in \mathbb{N}$.

(ii) *If* $k \geq 1$ *and* $n \geq 2$, *then*

$$\frac{G(k,n,m)}{B_n^{m+k-1}} \equiv \begin{cases} -B_{n-1} \pmod{B_n}, & \text{if } 2 \mid n \text{ and } 2 \mid k; \\ 1 \pmod{B_n}, & \text{otherwise.} \end{cases}$$

Proof. We prove (i) and (ii) by induction on k. Since $G(1,n,m) = B_n^m$, the result holds for $k=1$. If $k \geq 1$ and $B_n^{m+k-1} \mid G(k,n,m)$, then by Lemma 2.4 B_n^{m+k} divides $B_{nG(k,n,m)} = G(k+1,n,m)$. Therefore, (i) obtains. Next, we prove (ii). Let $n \geq 2$. It is easy to see that (ii) holds when $k=1$. Let $k \geq 2$ and assume that (ii) holds for $k-1$. Let $t = G(k-1,n,m)$. By Lemma 2.2, we obtain

$$G(k,n,m) = B_{nt} = \sum_{j=1}^{t} \binom{t}{j} B_n^j (-B_{n-1})^{t-j} B_j$$

By (i), we obtain $B_n^{m+k-2} \mid t$. By Lemma 2.3, $B_n^{m+k} \mid \binom{t}{j} B_n^j$, for $3 \leq j \leq t$. Therefore,

$$\binom{t}{j} B_n^{j-m-k+1} \equiv 0 \pmod{B_n} \quad \text{for } 3 \leq j \leq t.$$

Then

$$\frac{G(k,n,m)}{B_n^{m+k-1}} \equiv \sum_{j=1}^{2} \binom{t}{j} B_n^{j-m-k+1} (-B_{n-1})^{t-j} B_j \pmod{B_n}.$$

The corresponding term when $j=2$ in the above sum is

$$\frac{3t(t-1)}{B_n^{m+k-2}} B_n (-B_{n-1})^{t-2} \equiv 0 \pmod{B_n}.$$

Therefore,

$$\frac{G(k,n,m)}{B_n^{m+k-1}} \equiv \frac{t}{B_n^{m+k-2}} (-B_{n-1})^{t-1} \pmod{B_n}.$$

Next, we apply Lemmas 2.6 and 2.7 without further reference.

Case 1. n and k are even. Then t is even. By the inductive hypothesis, we obtain

$$\begin{aligned}\frac{G(k,n,m)}{B_n^{m+k-1}} &\equiv \frac{t}{B_n^{m+k-2}}(-B_{n-1})^{t-1} \equiv (-B_{n-1})^{t-1}\\ &\equiv -B_{n-1}B_{n-1}^{t-2} \equiv -B_{n-1} \pmod{B_n}.\end{aligned}$$

Case 2. n is even and k is odd. Then t is even. By the inductive hypothesis, we obtain

$$\begin{aligned}\frac{G(k,n,m)}{B_n^{m+k-1}} &\equiv \frac{t}{B_n^{m+k-2}}(-B_{n-1})^{t-1} \equiv (-B_{n-1})(-B_{n-1})^{t-1}\\ &\equiv B_{n-1}^{t} \equiv 1 \pmod{B_n}.\end{aligned}$$

Case 3. n is odd. Then t is odd. By the inductive hypothesis, we obtain

$$\frac{G(k,n,m)}{B_n^{m+k-1}} \equiv \frac{t}{B_n^{m+k-2}}(-B_{n-1})^{t-1} \equiv B_{n-1}^{t-1} \equiv 1 \pmod{B_n}.$$

This completes the proof. □

If $n \geq 2$, then $B_{n-1} \not\equiv 0 \pmod{B_n}$ and $1 \not\equiv 0 \pmod{B_n}$. By Theorem 3.1, it is easy to see $B_n^{m+k} \nmid G(k,n,m)$ if $n \geq 2$. Therefore, we obtain the following corollary:

Corollary 3.2. *Let $k, m \geq 1$ and $n \geq 2$. Then $B_n^{m+k-1} \parallel G(k,n,m)$.*

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