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γ-Paired Dominating Graphs of Paths

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Abstract

A set D of vertices in a graph $G = (V(G), E(G))$ is a paired dominating set of G if every vertex of G is adjacent to some vertex in D and the subgraph induced by D contains a perfect matching. The paired domination number of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of G. A paired dominating set of cardinality $\gamma_{pr}(G)$ is called a $\gamma_{pr}(G)$-set. The γ-paired dominating graph of G, denoted by $PD_\gamma(G)$, is the graph whose vertices are $\gamma_{pr}(G)$-sets and two $\gamma_{pr}(G)$-sets are adjacent in $PD_\gamma(G)$ if one can be obtained from the other by removing one vertex and adding another vertex of G. In this paper, we present the γ-paired dominating graphs of paths.

1 Introduction

For notation and terminology, we follow [8] in general. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the *degree* of v in G is the number of edges incident to v (each loop at v counts twice). A vertex of degree one is called a *leaf*, and its neighbor is a *support vertex*. The *neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$. We use $N[v]$ to denote the set $N(v) \cup \{v\}$. For a set $D \subseteq V(G)$, the vertices of

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D *dominate* the vertices in $S \subseteq V(G)$ if $S \subseteq N[D]$, where $N[D] = \bigcup_{v \in D} N[v]$. We use P_n and C_n to denote a *path* and a *cycle* with n vertices, respectively.

A set $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to a vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. A detailed literature on domination is provided by Haynes, Hedetniemi, and Slater [4, 5].

In 2010, Lakshmanan and Vijayakumar [6] defined the *gamma graph* $\gamma.G$ of a graph G as the graph whose vertices are $\gamma(G)$-sets, and $\gamma(G)$-sets D_1 and D_2 are adjacent in $\gamma.G$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$ and showed the gamma graphs of some simple graphs. In 2011, Fricke et al. [1] also defined the *gamma graph* $G(\gamma)$ with different meaning. The only difference is that two $\gamma(G)$-sets D_1 and D_2 are adjacent in $G(\gamma)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $uv \in E(G)$ and $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$.

In 2014, Haas and Seyffarth [2] introduced the k*-dominating graph* of G, denoted by $D_k(G)$. Its vertex set contains all dominating sets of size at most k. Two dominating sets D_1 and D_2 are adjacent in $D_k(G)$ if there exists a vertex $v \notin D_2$ of G such that $D_1 = D_2 \cup \{v\}$ and gave some conditions for connectivity of $D_k(G)$.

In 2017, Wongsriya and Trakultraipruk [9] introduced the γ*-total dominating graph* $TD_\gamma(G)$ of G as the graph whose vertices are minimum total dominating sets. Two minimum total dominating sets D_1 and D_2 are adjacent in $TD_\gamma(G)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$. The authors proved the results on the γ-total dominating graphs of paths and cycles. In 2019, Samanmoo et al. [7] defined the γ*-independent dominating graph* $ID_\gamma(G)$ of G as the graph whose vertices are minimum independent dominating sets. Two minimum independent dominating sets D_1 and D_2 are adjacent in $ID_\gamma(G)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$. The authors determined the γ-independent dominating graphs of paths and cycles.

A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M. Haynes and Slater [3] introduced the concept of paired domination. A set $D \subseteq V(G)$ is a *paired dominating set* of G if it is a dominating set of G, and the subgraph of G induced by D contains a perfect matching. For a paired dominating set D with a perfect matching M, the set $\{u, v\} \subseteq D$ is called *paired* if the edge $uv \in M$. The *paired domination number* of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of G. A paired dominating set of cardinality $\gamma_{pr}(G)$ is called a $\gamma_{pr}(G)$*-set*.

In this paper, we define the *γ-paired dominating graph* of a graph G, denoted by $PD_\gamma(G)$, as the graph whose vertices are $\gamma_{pr}(G)$-sets, and two $\gamma_{pr}(G)$-sets D_1 and D_2 are adjacent in $PD_\gamma(G)$ if there exist two vertices $u \in D_2$ and $v \notin D_2$ of G such that $D_1 = (D_2 \setminus \{u\}) \cup \{v\}$. In particular, we determine the γ-paired dominating graphs of paths. For instance, the γ-paired dominating graphs of the paths $P_5 : v_1v_2v_3v_4v_5$ and $P_6 : v_1v_2v_3v_4v_5v_6$ are shown in Figure 1. We see that $PD_\gamma(P_5) \cong C_3$ and $PD_\gamma(P_6) \cong C_4$.

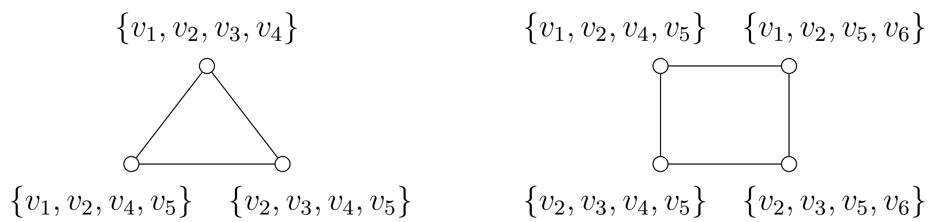

Figure 1: The γ-paired dominating graphs of P_5 and P_6, respectively

2 Preliminary Results

In this section, we first recall some known results which will be used in the proof of our main results.

Haynes and Slater [3] established the following two lemmas.

Lemma 2.1. *If v is a support vertex of a graph G, then v is in every paired dominating set of G.*

Lemma 2.2. *For any integer $n \geq 2$, $\gamma_{pr}(P_n) = 2\lceil \frac{n}{4} \rceil$.*

Lemma 2.3. *Let $n \geq 6$ be an integer. Then each $\gamma_{pr}(P_n)$-set cannot contain l consecutive vertices of P_n, where $6 \leq l \leq n$ is even.*

Proof. Let $P_n : v_1v_2v_3 \cdots v_n$ be the path. Suppose, to get a contradiction, that there is a $\gamma_{pr}(P_n)$-set D containing l consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+l-1}$, where $1 \leq i \leq n - l + 1$. Let D' be the set obtained from D by deleting two paired vertices from $v_{i+2}, v_{i+3}, \ldots, v_{i+l-3}$. Then D' is a paired dominating set with $|D'| < |D|$, a contradiction. $\square$

3 γ-Paired Dominating Graphs of Paths

In this section, we determine the γ-paired dominating graphs of paths. Let $P_n : v_1v_2v_3\cdots v_n$ be the path with n vertices. If $n = 1$, then there is no $\gamma_{pr}(P_1)$-set and so $PD_\gamma(P_1)$ is the null graph. For $n \geq 2$, we divide the value of n into the four cases as follows: $n \in \{4k, 4k-1, 4k-2, 4k-3\}$. If $n = 4k$, then we get the following result.

Theorem 3.1. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k}) \cong P_1$.*

Proof. By Lemma 2.2, $\gamma_{pr}(P_{4k}) = 2k$. This implies that any two adjacent vertices must dominate exactly four vertices in P_{4k}. Therefore, there is exactly one $\gamma_{pr}(P_{4k})$-set, which is $\{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, and thus $PD_\gamma(P_{4k}) \cong P_1$. □

Next, we consider the γ-paired dominating graph of a path with $4k-1$ vertices, where $k \geq 1$ is an integer. We first establish the following lemmas.

Lemma 3.2. *Let $k \geq 2$ be an integer. Then each $\gamma_{pr}(P_{4k-1})$-set cannot contain l consecutive vertices of P_{4k-1}, where $4 \leq l \leq 4k-2$ is even.*

Proof. Suppose, on the contrary, that there is a $\gamma_{pr}(P_{4k-1})$-set D containing l consecutive vertices of P_{4k-1}. Then these l vertices of D can dominate at most $l+2$ vertices in P_{4k-1}. Since $\gamma_{pr}(P_{4k-1}) = 2k$, the other $2k-l$ vertices of D must dominate at least $4k-1-(l+2) = 4k-l-3$ vertices in P_{4k-1}. Note that the $2k-l$ remaining vertices of D can dominate at most $4k-2l < 4k-l-3$ vertices in this path since $l \geq 4$. This means that D cannot dominate all vertices in P_{4k-1}, a contradiction. □

Lemma 3.3. *Let $k \geq 1$ be an integer. Then there is only one $\gamma_{pr}(P_{4k-1})$-set containing the pair $\{v_{4k-2}, v_{4k-1}\}$ and this set is of degree one in $PD_\gamma(P_{4k-1})$.*

Proof. It is clear that $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are the only $\gamma_{pr}(P_3)$-sets and so $PD_\gamma(P_3) \cong P_2$. The lemma holds for $k = 1$. Now let $k \geq 2$. We find all $\gamma_{pr}(P_{4k-1})$-sets containing the pair $\{v_{4k-2}, v_{4k-1}\}$. By Lemma 3.2, all such $\gamma_{pr}(P_{4k-1})$-sets cannot contain v_{4k-3}. The vertices v_{4k-2} and v_{4k-1} dominate $v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}$. By Lemma 2.2, $\gamma_{pr}(P_{4k-1}) = 2k$ and so the other $2k-2$ vertices must dominate all vertices in P_{4k-4}. Since $\gamma_{pr}(P_{4k-4}) = 2k-2$, these $2k-2$ vertices form a $\gamma_{pr}(P_{4k-4})$-set. By Theorem 3.1, $D = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-6}, v_{4k-5}\}$ is the only $\gamma_{pr}(P_{4k-4})$-set. Therefore, $D' = D \cup \{v_{4k-2}, v_{4k-1}\}$ is the only $\gamma_{pr}(P_{4k-1})$-set containing the pair $\{v_{4k-2}, v_{4k-1}\}$. It is clear that $(D' \setminus \{v_{4k-1}\}) \cup \{v_{4k-3}\}$ is a $\gamma_{pr}(P_{4k-1})$-set and it is the only neighbor of D' in $PD_\gamma(P_{4k-1})$. □

Theorem 3.4. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-1}) \cong P_{k+1}$.*

Proof. We prove this theorem by induction on k. We have showed that $PD_\gamma(P_3) \cong P_2$. Let $k \geq 2$ and suppose that $PD_\gamma(P_{4k-1}) \cong P_{k+1} : D_1D_2D_3 \cdots D_{k+1}$, where D_x is a $\gamma_{pr}(P_{4k-1})$-set for all $x \in \{1, 2, 3, \ldots, k+1\}$. Using Lemma 3.3, without loss of generality, we may assume that D_{k+1} contains the pair $\{v_{4k-2}, v_{4k-1}\}$. By Lemma 2.1, the vertex v_{4k-2} is in every $\gamma_{pr}(P_{4k-1})$-set. It follows that $D_1, D_2, D_3, \ldots, D_k$ contain the pair $\{v_{4k-3}, v_{4k-2}\}$.

We show that $PD_\gamma(P_{4k+3}) \cong P_{k+2}$. Note that each $\gamma_{pr}(P_{4k+3})$-set contains either the pair $\{v_{4k+1}, v_{4k+2}\}$ or the pair $\{v_{4k+2}, v_{4k+3}\}$. We first consider all $\gamma_{pr}(P_{4k+3})$-sets containing the pair $\{v_{4k+1}, v_{4k+2}\}$. By Lemma 3.2, all such $\gamma_{pr}(P_{4k+3})$-sets do not contain v_{4k}. Note that $\gamma_{pr}(P_{4k+3}) = 2k + 2$ and so the other $2k$ vertices must dominate all vertices in P_{4k-1}. Since $\gamma_{pr}(P_{4k-1}) = 2k$, these $2k$ vertices form a $\gamma_{pr}(P_{4k-1})$-set. Hence each $\gamma_{pr}(P_{4k+3})$-set containing the pair $\{v_{4k+1}, v_{4k+2}\}$ is a union of a $\gamma_{pr}(P_{4k-1})$-set and $\{v_{4k+1}, v_{4k+2}\}$. For each $x \in \{1, 2, 3, \ldots, k + 1\}$, let

$$D'_x = D_x \cup \{v_{4k+1}, v_{4k+2}\}.$$

By the inductive hypothesis, $D'_1, D'_2, D'_3, \ldots, D'_{k+1}$ are the only $\gamma_{pr}(P_{4k+3})$-sets containing the pair $\{v_{4k+1}, v_{4k+2}\}$ and they form a path with $k+1$ vertices in $PD_\gamma(P_{4k+3})$.

We next consider all $\gamma_{pr}(P_{4k+3})$-sets containing the pair $\{v_{4k+2}, v_{4k+3}\}$. Note that D'_{k+1} contains the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k+1}, v_{4k+2}\}$. Let

$$D'_{k+2} = (D'_{k+1} \setminus \{v_{4k+1}\}) \cup \{v_{4k+3}\}.$$

Lemma 3.3 shows that D'_{k+2} is the only $\gamma_{pr}(P_{4k+3})$-set containing the pair $\{v_{4k+2}, v_{4k+3}\}$ and it is adjacent only to D'_{k+1}. Hence, $D'_1, D'_2, D'_3, \ldots, D'_{k+2}$ are the only $\gamma_{pr}(P_{4k+3})$-sets and they form a path with $k + 2$ vertices. $\square$

Let G and H be graphs. The *Cartesian product* of G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ whose vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

Let $P_p : u_1u_2u_3 \cdots u_p$ and $P_q : v_1v_2v_3 \cdots v_q$, where p and q are positive integers. Fricke et al. [1] defined a *stepgrid* $SG_{p,q}$ to be the subgraph of $P_p \square P_q$ induced by $\{(u_x, v_y) : 1 \leq x \leq p, 1 \leq y \leq q, x - y \leq 1\}$. We call the vertex (u_x, v_y) in the stepgrid as the *vertex at the position* (x, y). For instance, the stepgrids $SG_{1,1}$ and $SG_{4,3}$ are shown in Figure 2.

Lemma 3.5. *Let $k \geq 3$ be an integer. Then all $\gamma_{pr}(P_{4k-2})$-sets containing the pair $\{v_{4k-3}, v_{4k-2}\}$ form a path $D'_1D'_2 \cdots D'_k$ with k vertices, where D'_1 and*

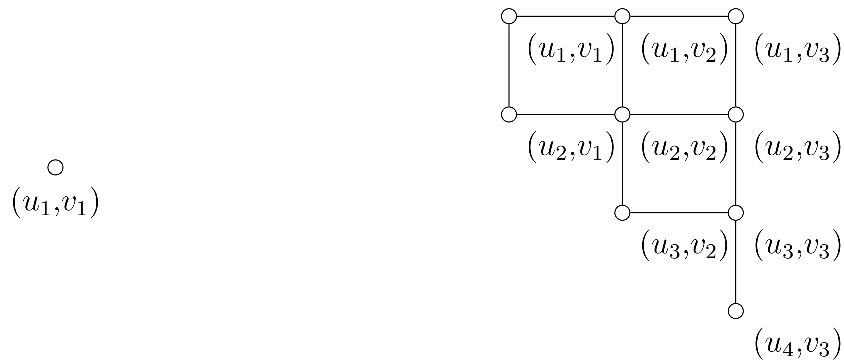

Figure 2: The stepgrids $SG_{1,1}$ and $SG_{4,3}$, respectively

D'_k are of degree two and the others are of degree three in $PD_\gamma(P_{4k-2})$. If D'_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, then the others contain the pair $\{v_{4k-7}, v_{4k-6}\}$. Moreover, D'_k has a neighbor of degree two in $PD_\gamma(P_{4k-2})$.

Proof. Similar to the proof of Lemma 3.2, we can prove that each $\gamma_{pr}(P_{4k-2})$-set containing the pair $\{v_{4k-3}, v_{4k-2}\}$ cannot contain the vertex v_{4k-4}. Note that such a $\gamma_{pr}(P_{4k-2})$-set is a union of a $\gamma_{pr}(P_{4k-5})$-set and $\{v_{4k-3}, v_{4k-2}\}$. By Theorem 3.4, $PD_\gamma(P_{4k-5}) \cong P_k : D_1 D_2 \cdots D_k$, where D_x is a $\gamma_{pr}(P_{4k-5})$-set for all $x \in \{1, 2, \ldots, k\}$. Using Lemma 3.3, without loss of generality, we may assume that D_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$ and the others contain the pair $\{v_{4k-7}, v_{4k-6}\}$. For each $x \in \{1, 2, \ldots, k\}$, let $D'_x = D_x \cup \{v_{4k-3}, v_{4k-2}\}$. Thus $D'_1, D'_2, \ldots, D'_k$ are the only $\gamma_{pr}(P_{4k-2})$-sets containing the pair $\{v_{4k-3}, v_{4k-2}\}$ and they also form a path with k vertices in $PD_\gamma(P_{4k-2})$. Now, D'_1 and D'_k have only one neighbor containing the pair $\{v_{4k-3}, v_{4k-2}\}$ and $D'_2, D'_3, \ldots, D'_{k-1}$ have two neighbors containing the pair $\{v_{4k-3}, v_{4k-2}\}$. Since each $\gamma_{pr}(P_{4k-2})$-set contains either the pair $\{v_{4k-4}, v_{4k-3}\}$ or the pair $\{v_{4k-3}, v_{4k-2}\}$, the other neighbors of D'_x must contain the pair $\{v_{4k-4}, v_{4k-3}\}$. Note that $D'_1, D'_2, \ldots, D'_{k-1}$ contain the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$ and D'_k contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$. Thus, for all $x \in \{1, 2, \ldots, k\}$, $D''_x = (D'_x \setminus \{v_{4k-2}\}) \cup \{v_{4k-4}\}$ is the only neighbor of D'_x containing the pair $\{v_{4k-4}, v_{4k-3}\}$. It follows that D'_1 and D'_k are of degree two and $D'_2, D'_3, \ldots, D'_{k-1}$ are of degree three in $PD_\gamma(P_{4k-2})$.

Finally, we show that D'_k has a neighbor of degree two, which is D''_k. Now, D''_k has only D'_k as the neighbor containing the pair $\{v_{4k-3}, v_{4k-2}\}$. Note that D''_k contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. Furthermore, we can easily prove that D''_k is the only $\gamma_{pr}(P_{4k-2})$-set containing these two pairs. Then we only need to find the neighbors of D''_k containing the pairs $\{v_{4k-7}, v_{4k-6}\}$,

$\{v_{4k-4}, v_{4k-3}\}$. Note that $D''_1, D''_2, \ldots, D''_{k-1}$ are the only $\gamma_{pr}(P_{4k-2})$-sets containing the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$ and $D''_1 D''_2 D''_3 \cdots D''_{k-1} D''_k$ is a path. Therefore, D''_k has only D''_{k-1} as the neighbor containing the pair $\{v_{4k-4}, v_{4k-3}\}$. We can conclude that D''_k is of degree two in $PD_\gamma(P_{4k-2})$. ☐

Theorem 3.6. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$.*

Proof. We proceed by induction on k. We have $PD_\gamma(P_2) \cong P_1 \cong SG_{1,1}$, $PD_\gamma(P_6) \cong C_4 \cong SG_{2,2}$ (see Figure 1), and $PD_\gamma(P_{10}) \cong SG_{3,3}$ shown in Figure 3, where $\{a, b, c, d, e, f\}$ stands for $\{v_a, v_b, v_c, v_d, v_e, v_f\}$. Let $k \geq 3$ and

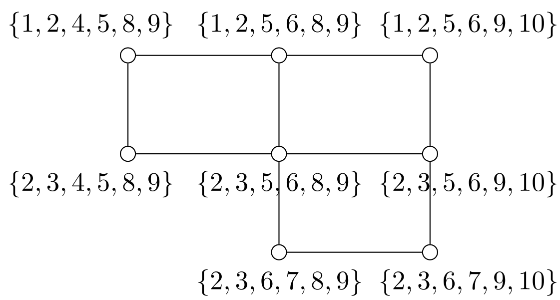

Figure 3: The γ-paired dominating graph of P_{10}

suppose that $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$. For all $x, y \in \{1, 2, \ldots, k\}$ with $x - y \leq 1$, let $D_{x,y}$ be the $\gamma_{pr}(P_{4k-2})$-set at the position (x, y) in $PD_\gamma(P_{4k-2})$ as shown in Figure 4. Using Lemma 3.5, without loss of generality, we may assume that $D_{1,k}, D_{2,k}, \ldots, D_{k-1,k}$ contain the pairs $\{v_{4k-7}, v_{4k-6}\}, \{v_{4k-3}, v_{4k-2}\}$ and $D_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-3}, v_{4k-2}\}$. Hence, $D_{x,y}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$, for all $y \neq k$.

We show that $PD_\gamma(P_{4k+2}) \cong SG_{k+1,k+1}$. Note that each $\gamma_{pr}(P_{4k+2})$-set must contain either the pair $\{v_{4k}, v_{4k+1}\}$ or the pair $\{v_{4k+1}, v_{4k+2}\}$. We first consider all $\gamma_{pr}(P_{4k+2})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$ but not the vertex v_{4k-1}. Note that such $\gamma_{pr}(P_{4k+2})$-set is a union of a $\gamma_{pr}(P_{4k-2})$-set and $\{v_{4k}, v_{4k+1}\}$. For all $x, y \in \{1, 2, \ldots, k\}$, let

$$D'_{x,y} = D_{x,y} \cup \{v_{4k}, v_{4k+1}\}.$$

By the induction hypothesis, all $D'_{x,y}$'s are the only $\gamma_{pr}(P_{4k+2})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1} and they form a stepgrid $SG_{k,k}$ in $PD_\gamma(P_{4k+2})$.

We next consider all $\gamma_{pr}(P_{4k+2})$-sets containing the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$. By Lemma 2.3, such $\gamma_{pr}(P_{4k+2})$-sets cannot contain the vertex v_{4k-3}. We can verify that there is exactly one $\gamma_{pr}(P_{4k+2})$-set containing

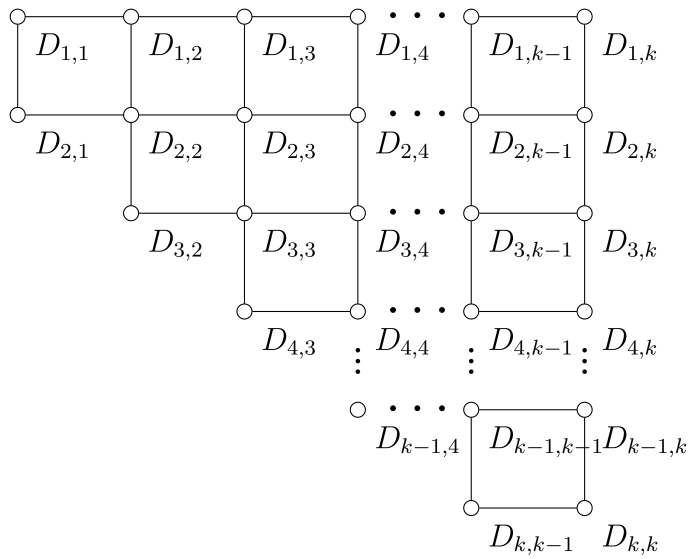

Figure 4: The γ-paired dominating graph of P_{4k-2}

the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$. Note that $D'_{1,k}, D'_{2,k}, \ldots, D'_{k-1,k}$ contain the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$ and $D'_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-3}, v_{4k-2}\}, \{v_{4k}, v_{4k+1}\}$. We see that $(D'_{x,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k-1}\}$ is a dominating set if and only if $x = k$. Then we let

$$D'_{k+1,k} = (D'_{k,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k-1}\}.$$

Hence $D'_{k+1,k}$ is the only $\gamma_{pr}(P_{4k+2})$-set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$ and it is adjacent to $D'_{k,k}$. Clearly, $D'_{k+1,k}$ is not adjacent to any $\gamma_{pr}(P_{4k+2})$-set containing the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, which is $D'_{x,y}$ with $y \neq k$.

Finally, we consider all $\gamma_{pr}(P_{4k+2})$-sets containing the pair $\{v_{4k+1}, v_{4k+2}\}$. Note that $D'_{1,k}, D'_{2,k}, \ldots, D'_{k,k}$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}, \{v_{4k}, v_{4k+1}\}$ and $D'_{k+1,k}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}, \{v_{4k}, v_{4k+1}\}$. For each $x \in \{1, 2, \ldots, k+1\}$, let

$$D'_{x,k+1} = (D'_{x,k} \setminus \{v_{4k}\}) \cup \{v_{4k+2}\}.$$

By Lemma 3.5, $D'_{1,k+1}, D'_{2,k+1}, \ldots, D'_{k+1,k+1}$ are the only $\gamma_{pr}(P_{4k+2})$-sets containing the pair $\{v_{4k+1}, v_{4k+2}\}$ and they form a path with $k+1$ vertices. Also, $D'_{x,k+1}$ is adjacent to $D'_{x,k}$ for all $x \in \{1, 2, \ldots, k+1\}$. Now, $D'_{1,k+1}$ and $D'_{k+1,k+1}$ already have degree two, and the others have degree three.

Therefore, all $\gamma_{pr}(P_{4k+2})$-sets form a stepgrid $SG_{k+1,k+1}$. This completes the proof. $\square$

Let $P_p : u_1u_2u_3\cdots u_p$, $P_q : v_1v_2v_3\cdots v_q$, and $P_r : w_1w_2w_3\cdots w_r$, where p, q, and r are positive integers. We define a *stepgrid* $SG_{p,q,r}$ be the graph with vertex set

$$V(SG_{p,q,r}) = \{(u_x, v_y, w_z) \in V(P_p\Box P_q\Box P_r) : 1 \leq x \leq p, 1 \leq y \leq q, 1 \leq z \leq r, x-y \leq 0, x-z \leq 1, y-z \geq 0\},$$

and edge set

$$E(SG_{p,q,r}) = E(P_p\Box P_q\Box P_r) \cup \{(u_x, v_x, w_x)(u_{x+1}, v_{x+1}, w_x) : 1 \leq x \leq p-1\}.$$

The vertex (u_x, v_y, w_z) is called the *vertex at the position* (x, y, z) in $SG_{p,q,r}$. For example, the stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$ are shown in Figures 5 and 6, respectively.

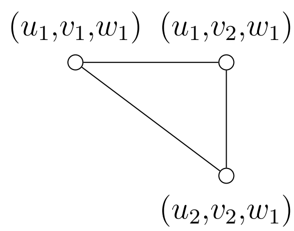

Figure 5: The stepgrid $SG_{2,2,1}$

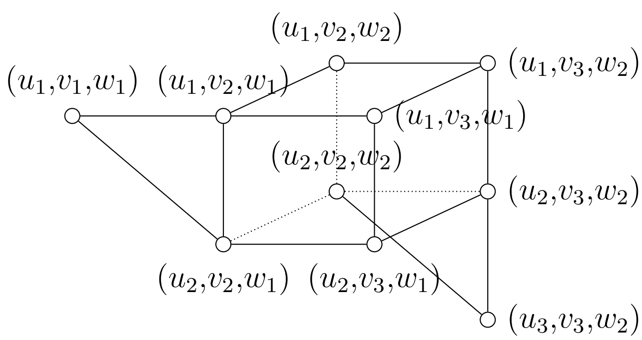

Figure 6: The stepgrid $SG_{3,3,2}$

Lemma 3.7. *Let $k \geq 3$ be an integer. Then all $\gamma_{pr}(P_{4k-3})$-sets containing the pair $\{v_{4k-4}, v_{4k-3}\}$ form a stepgrid $SG_{k,k-1}$ (see Figure 7), where $D'_{1,1}, D'_{2,1}, D'_{1,k-1}$ are of degree three, $D'_{2,k-1}, D'_{3,k-1}, \ldots, D'_{k-1,k-1}$ are of degree four, and $D'_{k,k-1}$ is of degree two in $PD_\gamma(P_{4k-3})$. Moreover, $D'_{x,k-1}$ contains the pair $\{v_{4k-7}, v_{4k-6}\}$ for all $x \in \{1, 2, \ldots, k-1\}$ and $D'_{k,k-1}$ contains the pair $\{v_{4k-6}, v_{4k-5}\}$.*

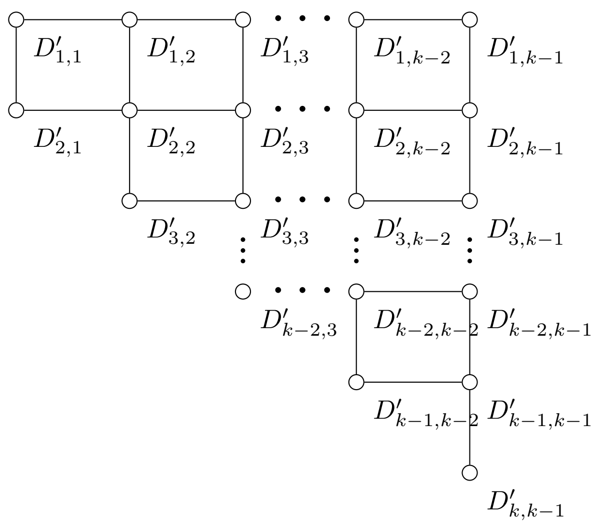

Figure 7: The stepgrid $SG_{k,k-1}$

Proof. We first consider all $\gamma_{pr}(P_{4k-3})$-sets containing $\{v_{4k-4}, v_{4k-3}\}$ but not the vertex v_{4k-5}. Then such a $\gamma_{pr}(P_{4k-3})$-set is a union of a $\gamma_{pr}(P_{4k-6})$-set and $\{v_{4k-4}, v_{4k-3}\}$. By Theorem 3.6, we have $PD_\gamma(P_{4k-6}) \cong SG_{k-1,k-1}$. For all $x, y \in \{1, 2, \ldots, k-1\}$ with $x - y \leq 1$, let $D_{x,y}$ be the $\gamma_{pr}(P_{4k-6})$-set at the position (x, y) in $PD_\gamma(P_{4k-6})$. Using Lemma 3.5, without loss of generality, we may assume that $D_{1,k-1}, D_{2,k-1}, \ldots, D_{k-2,k-1}$ contain the pairs $\{v_{4k-11}, v_{4k-10}\}$, $\{v_{4k-7}, v_{4k-6}\}$ and $D_{k-1,k-1}$ contains the pairs $\{v_{4k-10}, v_{4k-9}\}$, $\{v_{4k-7}, v_{4k-6}\}$. Then the other $\gamma_{pr}(P_{4k-3})$-sets contain the pair $\{v_{4k-8}, v_{4k-7}\}$. For all $x, y \in \{1, 2, \ldots, k-1\}$, let $D'_{x,y} = D_{x,y} \cup \{v_{4k-4}, v_{4k-3}\}$. These $D'_{x,y}$'s are the only $\gamma_{pr}(P_{4k-3})$-sets containing $\{v_{4k-4}, v_{4k-3}\}$ but not v_{4k-5}, and they also form a stepgrid $SG_{k-1,k-1}$ in $PD_\gamma(P_{4k-3})$. Then $D'_{x,k-1}$ contains the pair $\{v_{4k-7}, v_{4k-6}\}$ for all $x \in \{1, 2, \ldots, k-1\}$.

We next consider all $\gamma_{pr}(P_{4k-3})$-sets containing the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. Lemma 2.3 implies that such $\gamma_{pr}(P_{4k-3})$-set is a union of a

$\gamma_{pr}(P_{4k-8})$-set and these two pairs. Theorem 3.1 shows that a $\gamma_{pr}(P_{4k-3})$-set containing the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$ is unique. Note that $D'_{1,k-1}, D'_{2,k-1}, \ldots, D'_{k-2,k-1}$ contain the pairs $\{v_{4k-11}, v_{4k-10}\}$, $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$ and $D'_{k-1,k-1}$ contains the pairs $\{v_{4k-10}, v_{4k-9}\}$, $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$. Then $(D'_{x,k-1} \setminus \{v_{4k-7}\}) \cup \{v_{4k-5}\}$ is a dominating set if and only if $x = k-1$. Let $D'_{k,k-1} = (D'_{k-1,k-1} \setminus \{v_{4k-7}\}) \cup \{v_{4k-5}\}$. Then $D'_{k,k-1}$ is the only $\gamma_{pr}(P_{4k-3})$-set containing the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$ and it is adjacent to $D'_{k-1,k-1}$. Since all $D'_{x,y}$'s with $y \neq k-1$ contain the pairs $\{v_{4k-8}, v_{4k-7}\}$, $\{v_{4k-4}, v_{4k-3}\}$, the set $D'_{k,k-1}$ is not adjacent to them. Therefore, all $\gamma_{pr}(P_{4k-3})$-sets containing the pair $\{v_{4k-4}, v_{4k-3}\}$ form a stepgrid $SG_{k,k-1}$ in $PD_\gamma(P_{4k-3})$.

Note that each $\gamma_{pr}(P_{4k-3})$-set must contain either the pair $\{v_{4k-5}, v_{4k-4}\}$ or the pair $\{v_{4k-4}, v_{4k-3}\}$. Then for $x, y \in \{1, 2, \ldots, k-1\}$, $(D'_{x,y} \setminus \{v_{4k-3}\}) \cup \{v_{4k-5}\}$ is the only neighbor of $D'_{x,y}$ containing the pair $\{v_{4k-5}, v_{4k-4}\}$. Thus, $D'_{1,1}, D'_{2,1}, D'_{1,k-1}$ are of degree three and $D'_{2,k-1}, D'_{3,k-1}, \ldots, D'_{k-1,k-1}$ are of degree four in $PD_\gamma(P_{4k-3})$. Notice that $D'_{k,k-1}$ contains the pairs $\{v_{4k-10}, v_{4k-9}\}$, $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. Thus, $(D'_{k,k-1} \setminus \{v_{4k-3}\}) \cup \{v_{4k-7}\}$ is the only neighbor of $D'_{k,k-1}$ containing the pair $\{v_{4k-5}, v_{4k-4}\}$, so $D'_{k,k-1}$ is of degree two in $PD_\gamma(P_{4k-3})$. □

Theorem 3.8. *Let $k \geq 2$ be an integer. Then $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$.*

Proof. We prove this theorem by induction on k. We have $PD_\gamma(P_5) \cong C_3 \cong SG_{2,2,1}$ (see Figure 1). Figure 8 shows $PD_\gamma(P_9) \cong SG_{3,3,2}$, where we write $\{a, b, c, d, e, f\}$ for $\{v_a, v_b, v_c, v_d, v_e, v_f\}$. Let $k \geq 3$ and suppose that

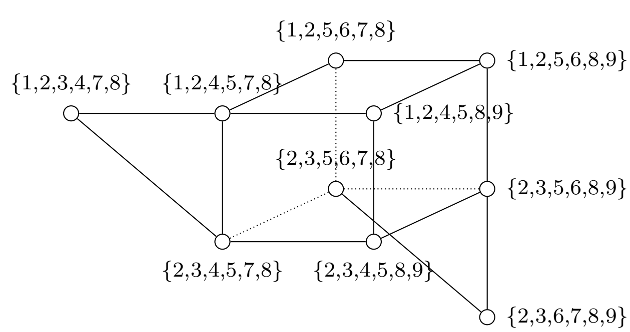

Figure 8: The γ-paired dominating graph of P_9

$PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$. For all $x, y \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$ with $x - y \leq 0$, $x - z \leq 1$ and $y - z \geq 0$, let $D_{x,y,z}$ be the $\gamma_{pr}(P_{4k-3})$-set at the position (x, y, z) in $PD_\gamma(P_{4k-3})$ as shown in Figure 9. Using

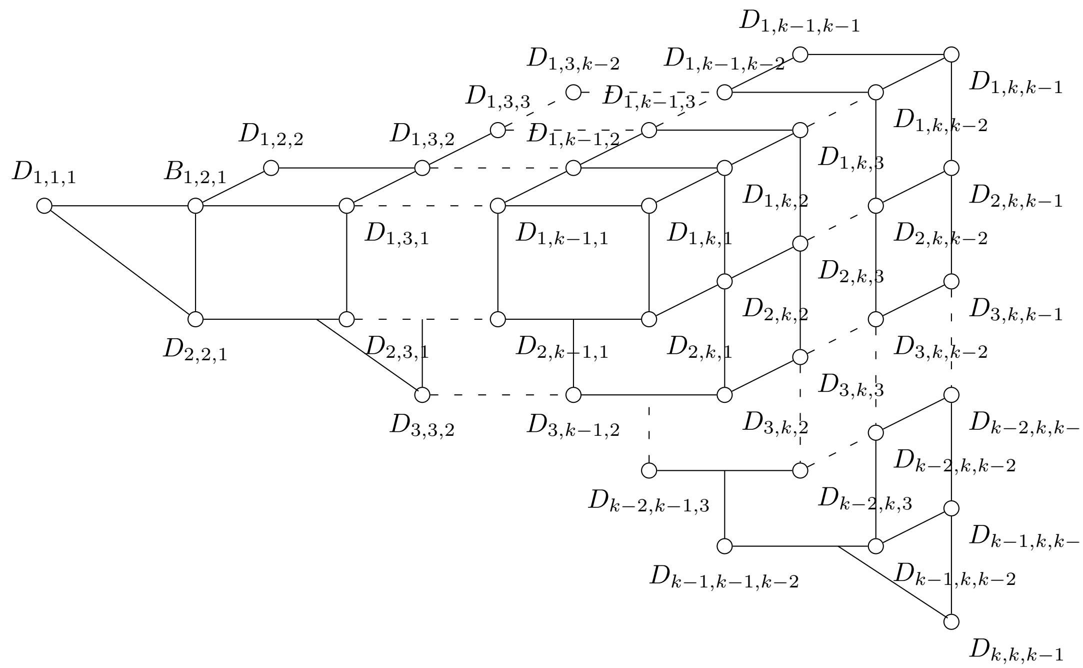

Figure 9: The γ-paired dominating graph of P_{4k-3}

Lemma 3.7, without loss of generality, we may assume that for all $x \in \{1, 2, \dots, k\}$, $z \in \{1, 2, \dots, k-1\}$, $D_{x,k,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$ and $D_{x,y,z}$ contains the pair $\{v_{4k-5}, v_{4k-4}\}$ for all $y \neq k$. In particular, $D_{1,k,k-1}, D_{2,k,k-1}, \dots, D_{k-1,k,k-1}$ contain the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$ and $D_{k,k,k-1}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. Hence $D_{x,k,z}$ contains the pairs $\{v_{4k-8}, v_{4k-7}\}$, $\{v_{4k-4}, v_{4k-3}\}$ for all $z \neq k-1$.

We show that $PD_\gamma(P_{4k+1}) \cong SG_{k+1,k+1,k}$. Note that each $\gamma_{pr}(P_{4k+1})$-set must contain either the pair $\{v_{4k-1}, v_{4k}\}$ or the pair $\{v_{4k}, v_{4k+1}\}$. We first consider all $\gamma_{pr}(P_{4k+1})$-sets containing the pair $\{v_{4k-1}, v_{4k}\}$ but not the vertex v_{4k-2}. Note that such a $\gamma_{pr}(P_{4k+1})$-set is a union of a $\gamma_{pr}(P_{4k-3})$-set and $\{v_{4k-1}, v_{4k}\}$. For all $x, y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k-1\}$, let

$$D'_{x,y,z} = D_{x,y,z} \cup \{v_{4k-1}, v_{4k}\}.$$

By the inductive hypothesis, all $D'_{x,y,z}$'s are the only $\gamma_{pr}(P_{4k+1})$-sets containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, and they form a stepgrid $SG_{k,k,k-1}$ in $PD_\gamma(P_{4k+1})$.

We then consider all $\gamma_{pr}(P_{4k+1})$-sets containing the pairs $\{v_{4k-3}, v_{4k-2}\}$ and $\{v_{4k-1}, v_{4k}\}$. By Lemma 2.3, such $\gamma_{pr}(P_{4k+1})$-sets cannot contain the vertex v_{4k-4}. Then each such $\gamma_{pr}(P_{4k+1})$-set is a union of a $\gamma_{pr}(P_{4k-5})$-set and $\{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\}$. Theorem 3.4 implies that there are k $\gamma_{pr}(P_{4k+1})$-sets containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$. Recall that if $z \neq k-1$, then $D'_{x,k,z}$ contains the pairs $\{v_{4k-8}, v_{4k-7}\}$, $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$ and hence $(D'_{x,k,z} \setminus \{v_{4k-4}\}) \cup \{v_{4k-2}\}$ is not a dominating set. Also, $D'_{1,k,k-1}, D'_{2,k,k-1}, \ldots, D'_{k-1,k,k-1}$ contain the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$ and $D'_{k,k,k-1}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$. For each $x \in \{1, 2, \ldots, k\}$, let

$$D'_{x,k,k} = (D'_{x,k,k-1} \setminus \{v_{4k-4}\}) \cup \{v_{4k-2}\}.$$

These $D'_{x,k,k}$'s are the only $\gamma_{pr}(P_{4k+1})$-sets containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$. Also, for all $x \in \{1, 2, \ldots, k\}$, $D'_{x,k,k}$ is adjacent to $D'_{x,k,k-1}$, but it is not adjacent to any $\gamma_{pr}(P_{4k+1})$-set containing the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, which is $D'_{x,y,z}$ with $y \neq k$.

We finally consider all $\gamma_{pr}(P_{4k+1})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$. Recall that for all $x \in \{1, 2, \ldots, k\}$, $z \in \{1, 2, \ldots, k-1\}$, $D'_{x,k,z}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$ and $D'_{x,k,k}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$. For each $x, z \in \{1, 2, \ldots, k\}$, let

$$D'_{x,k+1,z} = (D'_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{v_{4k+1}\}.$$

Then $D'_{x,k+1,z}$ is a $\gamma_{pr}(P_{4k+1})$-set containing the pair $\{v_{4k}, v_{4k+1}\}$ and it is adjacent to $D'_{x,k,z}$. Note that $D'_{k,k+1,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$. Let

$$\begin{aligned} D'_{k+1,k+1,k} &= (D'_{k,k+1,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k-1}\} \\ &= (D'_{k,k,k} \setminus \{v_{4k-1}\} \cup \{v_{4k+1}\}) \setminus \{v_{4k-3}\} \cup \{v_{4k-1}\} \\ &= (D'_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k+1}\}. \end{aligned}$$

Thus $D'_{k+1,k+1,k}$ is a $\gamma_{pr}(P_{4k+1})$-set containing the pair $\{v_{4k}, v_{4k+1}\}$ and it is adjacent to both $D'_{k,k+1,k}$ and $D'_{k,k,k}$. By Lemma 3.7, for all $x \in \{1, 2, \ldots, k+1\}$, $z \in \{1, 2, \ldots, k\}$, these $D'_{x,k+1,z}$'s are the only $\gamma_{pr}(P_{4k+1})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$ and they form a stepgrid $SG_{k+1,k}$ in $PD_{\gamma}(P_{4k+1})$. Moreover, the degree of each $D'_{x,k+1,z}$ satisfies Lemma 3.7. Now, all $\gamma_{pr}(P_{4k+1})$-sets form a stepgrid $SG_{k+1,k+1,k}$, so the theorem follows. □

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