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A simple proof of a relationship among the Zeta, Polygamma, and Clausen functions for the case $\pi/2$

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Abstract

We present expressions that relate the Riemann Zeta function, the Polygamma function in the case 1/4 as well as the Clausen functions for $\pi/2$, having been obtained by means of elementary procedures from the function $\cos(nx)/n^{m-1}$.

1 Introduction

Diversity of relationships are known for the Gamma, Digamma, and Polygamma functions, the Zeta and Hurwitz functions, the Dirichlet Beta function, the generalized harmonic numbers and the Clausen generalized functions, or the Barnes G function, among others [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16]. We will give an elementary proof of a relationship among the Zeta, Polygamma, and Clausen functions in the special case $z = \frac{\pi}{2}$.

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2 Expression for $\zeta(m)$, $\Psi^{(m-1)}(1/4)$ and $S_m(\frac{\pi}{2})$

We will base our reasoning on a function and its displacement by the value $\frac{\pi}{2}$; that is, $\frac{\cos(nx)}{n^{m-1}}$ and $\frac{\cos(nx-\frac{\pi}{2})}{n^{m-1}}$.

For all $m \in \mathbb{R}$, $m \neq 0$, the following equality

$$\sum_{n=1}^{\infty}\left(\int_0^{\frac{\pi}{2}} \frac{\cos(nx)}{n^{m-1}}dx + \int_0^{\frac{\pi}{2}} \frac{\cos(nx-\frac{\pi}{2})}{n^{m-1}}dx\right) = \\ = \sum_{n=1}^{\infty}\left(2\int_0^{\frac{\pi}{2}} \frac{\cos((4n-3)x)}{(4n-3)^{m-1}}dx + \int_0^{\frac{\pi}{2}} \frac{\cos((4n-2)x-\frac{\pi}{2})}{(4n-2)^{m-1}}dx\right) \tag{2.1}$$

follows because, in the cases where $n = \dot{4}$, both the initial function and its displaced cancel out; if $n = 1\,mod\,4$, both match; if $n = 3\,mod\,4$, their values are opposite; in cases where $n = \dot{2}$, the shifted function is canceled.

Let us calculate the value of each of the four addends of the equation (2.1):

1. In this first addend, we have

$$\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}} \frac{\cos(nx)}{n^{m-1}}dx = \sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^m}.$$

2. It follows that

$$\int_0^{\frac{\pi}{2}} \frac{\cos(nx-\frac{\pi}{2})}{n^{m-1}}dx = \frac{1}{n^m} - \frac{\cos(n\frac{\pi}{2})}{n^m}. \tag{2.2}$$

Applying the summation to (2.2), the second part of the equality is:

$$\zeta(m) - \sum_{n=1}^{\infty}\frac{\cos(n\frac{\pi}{2})}{n^m} = \zeta(m) - \left(-\frac{1}{2^m} + \frac{1}{4^m} - \frac{1}{6^m} + \frac{1}{8^m} \pm ...\right) = \\ = \zeta(m) - \left(-\left(\frac{1}{2^m} + \frac{1}{6^m} + \frac{1}{10^m} + ...\right) + \left(\frac{1}{4^m} + \frac{1}{8^m} + \frac{1}{12^m} + ...\right)\right).$$

It is immediate to see that

$$-\left(\frac{1}{2^m}+\frac{1}{6^m}+...\right)=-\sum_{n=1}^{\infty}\frac{1}{(4n-2)^m}=-\frac{1}{2^m}\left(\frac{1}{1^m}+\frac{1}{3^m}+...\right)=$$
$$=-\frac{1}{2^m}\left(\zeta(m)-\left(\frac{1}{2^m}+\frac{1}{4^m}+...\right)\right)=-\frac{1}{2^m}\left(\zeta(m)-\frac{1}{2^m}\zeta(m)\right)=$$
$$=-\frac{2^m-1}{2^m}\cdot\frac{\zeta(m)}{2^m}.$$

But

$$\left(\frac{1}{4^m}+\frac{1}{8^m}+\frac{1}{12^m}+...\right)=\frac{1}{4^m}\cdot\zeta(m).$$

We then have:

$$\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{\cos(nx-\frac{\pi}{2})}{n^{m-1}}dx=$$
$$=\left(\frac{-1+2^m}{2^m}\frac{1}{2^m}-\frac{1}{4^m}+1\right)\cdot\zeta(m)=(1+2^{-m}-2^{1-2m})\cdot\zeta(m).$$

3. Clearly,

$$\sum_{n=1}^{\infty}2\int_0^{\frac{\pi}{2}}\frac{\cos((4n-3)x)}{(4n-3)^{m-1}}dx=\sum_{n=1}^{\infty}\frac{2}{(4n-3)^m}=2\left(1+\frac{1}{5^m}+\frac{1}{9^m}+...\right). \tag{2.3}$$

4. This fourth addend is of the form

$$\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{\cos((4n-2)x-\frac{\pi}{2})}{(4n-2)^{m-1}}dx=\sum_{n=1}^{\infty}\frac{2}{(4n-2)^m}=$$
$$=2\left(\frac{1}{2^m}+\frac{1}{6^m}+\frac{1}{10^m}+...\right)=2\left(\frac{1}{2^m}+\frac{1}{2^m3^m}+\frac{1}{2^m5^m}+...\right)=$$
$$=\frac{1}{2^{m-1}}\left(1+\frac{1}{3^m}+\frac{1}{5^m}+...\right)=$$
$$=\frac{1}{2^{m-1}}\left(\zeta(m)-\left(\frac{1}{2^m}+\frac{1}{4^m}+\frac{1}{6^m}+...\right)\right)=$$
$$=\frac{1}{2^{m-1}}\left(\zeta(m)-\frac{1}{2^m}\zeta(m)\right)=\frac{2^m-1}{2^{2m-1}}\cdot\zeta(m).$$

As a result, we have:

$$\left(\sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^m}\right)+(1+2^{-m}-2^{1-2m})\cdot\zeta(m)=$$
$$=\left(2\sum_{n=1}^{\infty}\frac{1}{(4n-3)^m}\right)+\frac{2^m-1}{2^{2m-1}}\cdot\zeta(m).$$

Solving for $\zeta(m)$, we get:

$$\zeta(m)=\frac{2\sum_{n=1}^{\infty}\frac{1}{(4n-3)^m}-\sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^m}}{1-\frac{1}{2^m}}. \tag{2.4}$$

Recalling the Digamma and Polygamma functions:

$$\Psi(z)=\frac{\frac{\mathrm{d}\Gamma(z)}{\mathrm{d}z}}{\Gamma(z)},\qquad \Psi^{(n)}(z)=\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Psi(z),\ z\in\mathbb{C},\ n\in\mathbb{N},$$

and using the relation [1]:

$$\Psi^{(m)}(z)=(-1)^{m+1}\cdot m!\cdot\sum_{n=0}^{\infty}\frac{1}{(z+n)^{m+1}},\ z\in\mathbb{C},$$

we can express (2.3) as

$$\sum_{n=1}^{\infty}\frac{2}{(4n-3)^m}=\frac{2}{4^m}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{3}{4})^m}=$$
$$=\frac{2}{4^m}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{4})^m}=\frac{2\cdot(-1)^m\cdot\Psi^{(m-1)}(1/4)}{4^m\cdot(m-1)!}.$$

Therefore, (2.4) takes the form

$$\zeta(m)=\frac{\frac{2\cdot(-1)^m\cdot\Psi^{(m-1)}(1/4)}{4^m\cdot(m-1)!}-\sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^m}}{1-\frac{1}{2^m}}.$$

Taking into account definition [10, 15]:

$$S_m(x)=\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^m},$$

and considering the generalized Clausen function given by:

$$Cl_m(x) = \begin{cases} S_m(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^m} & , \quad m = \dot{2} \\ C_m(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^m} & , \quad m \neq \dot{2} \end{cases}$$

and

$$Gl_m(x) = \begin{cases} S_m(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^m} & , \quad m \neq \dot{2} \\ C_m(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^m} & , \quad m = \dot{2} \end{cases}$$

taking $x = \dfrac{\pi}{2}$, we can give the general expression

$$\zeta(m) = \frac{\frac{2\cdot(-1)^m\cdot\Psi^{(m-1)}(1/4)}{4^m\cdot(m-1)!} - S_m(\frac{\pi}{2})}{1 - \frac{1}{2^m}}.$$

Solving for $S_m(\frac{\pi}{2})$, we get:

$$S_m(\frac{\pi}{2}) = \left(\frac{1}{2^m} - 1\right)\zeta(m) + \frac{(-1)^m \cdot 2^{1-2m}}{(m-1)!}\Psi^{(m-1)}(\frac{1}{4}),$$

where in the cases of even m, we obtain:

$$Cl_m(\frac{\pi}{2}) = \left(\frac{1}{2^m} - 1\right)\zeta(m) + \frac{(-1)^m \cdot 2^{1-2m}}{(m-1)!}\Psi^{(m-1)}(\frac{1}{4}), \quad m = \dot{2}$$

and in the cases of odd m, we obtain:

$$Gl_m(\frac{\pi}{2}) = \left(\frac{1}{2^m} - 1\right)\zeta(m) + \frac{(-1)^m \cdot 2^{1-2m}}{(m-1)!}\Psi^{(m-1)}(\frac{1}{4}), \quad m \neq \dot{2}.$$

Notice that, in the cases in which $m = \dot{2}$, the value of $\zeta(m)$ has the expression of a monomial of degree m, $M_{\mathbb{Q}}(\pi)$, of rational coefficient in π and in the cases where $m \neq \dot{2}$, the one that would have a monomial structure of degree m, $M_{\mathbb{Q}}(\pi)$, would be $S_m(x)$.

References

[1] Milton Abramowitz, Irene A. Stegun, Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables, Dover Publications: New York, 1965.

[2] Victor S. Adamchik, Polygamma Functions of Negative Order, J. Comput. Appl. Math., **100,** (1998), 191–199.

[3] Victor S. Adamchik, On the Hurwitz Function for Rational Arguments, Appl. Math. Comput., **187,** (2007), 3–12.

[4] Horst Alzer, Jim Wells, Inequalities for the Polygamma Functions, SIAM J. Math. Anal., **29,** no. 6, (1998), 1459–1466.

[5] George E. Andrews, Richard Askey, Ranjan Roy, Special Functions. Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, 1999.

[6] George B. Arfken, Digamma and Polygamma Functions. In: Mathematical Methods for Physicists (George B. Arfken, Hans J. Weber, Frank E. Harris, editors), Academic Press, Elsevier, Amsterdam, 2013, 610–616.

[7] Jonathan M. Borwein, David J. Broadhurst, Joel Kamnitzer, Central Binomial Sums, Multiple Clausen Values and Zeta Values, Experimental Math., **10,** no. 1, (2001), 25–34.

[8] Junesang Choi, Djurdje Cvijovic, Values of the Polygamma Functions at Rational Arguments, J. Phys. A., **40,** (2007), 15019–15028.

[9] Junesang Choi, Djurdje Cvijovic, (Corrigendum) Values of the Polygamma Functions at Rational Arguments, J. Phys. A., **43,** (2010), 239801.

[10] Thomas Clausen, Über die Function $\sin\varphi + \frac{1}{2^2}\sin 2\varphi + \frac{1}{3^2}\sin 3\varphi$ +etc., J. Reine Angew. Math., **8,** (1832), 298–300.

[11] Mark W. Coffey, On One-Dimensional Digamma and Polygamma Series Related to the Evaluation of Feynman Diagrams, J. Comput. Appl. Math., **183,** (2005), 84–100.

[12] Olivier Espinosa, and Victor H. Moll, A Generalized Polygamma Function, Integral Trans. Special Func., **15,** (2004), 101–115.

[13] Kurt S. Kölbig, The Polygamma Function and the Derivatives of the Cotangent Function for Rational Arguments, CERN-IT-Reports, CERN-CN-96-005, (1996), 1–14.

[14] Kurt S. Kölbig, The Polygamma Function $\psi^{(k)}(x)$ for $x = 1/4$ and $x = 3/4$, J. Comput. Appl. Math., **75,** no. 1, (1996), 43–46.

[15] Leonard Lewin, Polylogarithms and Associated Functions, North-Holland, New York, 1981.

[16] Edmund T. Whittaker, George N. Watson, A Course of Modern Analysis, An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions, 3rd ed., Cambridge University Press, 1920.