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$w(k,l)$-semi-open sets in bi-weak structure spaces

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Abstract

In this article, we deal with the concept of $w(k,l)$-semi-open sets in bi-weak structure spaces, where $k,l \in \{1,2\}$ with $k \neq l$. We obtain some properties of $w(k,l)$-semi-open sets. Finally, we introduce the notion of $w(k,l)$-semi-continuity.

1 Introduction

In 2000, Popa and Noiri [1] introduced the concept of the minimal structure. In 2002, Császár [2] introduced the concept of the generalized topology. Later, in 2011, Császár [3] introduced the concept of the weak structure. Such structures are some generalizations of topology. As a result, they have expanded the concepts of semi-open sets, α-open sets, β-open sets and preopen sets, especially in the weak structure. Császár [3] used the symbol $\sigma(w)$, $\alpha(w)$, $\beta(w)$ and $\pi(w)$ to represent the classes of semi-open sets, α-open sets, β-open sets and preopen sets in the weak structure, respectively. And he also showed that these classes a generalized topology. In 2017, Puiwong et al [4] introduced a new space, consisting of a nonempty set X and two weak structures w^1, w^2 on X, called the bi-weak structure space. Also, they studied open sets, closed sets, and separation axioms on this space.

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It is therefore natural for us to expand the concept of semi-open set in a topological space to a bi-weak structure space which is called a $w(k,l)$-semi-open set in a bi-weak structure space, where $k,l \in \{1,2\}$ with $k \neq l$. Moreover, we study some properties of this set. Furthermore, we introduce the notion of continuity using $w(k,l)$-semi-open sets.

2 Preliminaries

In this section, we shall recall some basic definitions and notations of weak structure and bi-weak structure space following [3] and [4].

Let X be a nonempty set and let $P(X)$ the power set of X. A subfamily w of $P(X)$ is called a *weak structure* (briefly *WS*) on X iff $\emptyset \in w$. The elements of w are called *w-open sets* and the complements are called *w-closed sets*. For a *WS* w on X and a subset A of X, we define $i_w(A)$ as the union of all w-open sets contained in A and $c_w(A)$ as the intersection of all w-closed sets containing A. Obviously, $i_w(A) \subset A \subset c_w(A)$. It is easy to verify that i_w and c_w are idempotent (i.e., if $A \subset X$, then $i_w(i_w(A)) = i_w(A)$ and $c_w(c_w(A)) = c_w(A)$) and monotonic (i.e., if $A \subset B \subset X$, then $i_w(A) \subset i_w(B)$ and $c_w(A) \subset c_w(B)$). Moreover, $i_w(X - A) = X - c_w(A)$ and $c_w(X - A) = X - i_w(A)$. If $A \in w$, then $A = i_w(A)$ and if A is w-closed, then $A = c_w(A)$. It is well known that $x \in c_w(A)$ if and only if $U \cap A \neq \emptyset$, for all w-open set U containing x. Furthermore, $x \in i_w(A)$ if and only if there is $U \in w$ such that $x \in U \subset A$.

Let X be a nonempty set and let w^1, w^2 be two weak structures on X. A triple (X, w^1, w^2) is called a *bi-weak structure space* (briefly *bi-w space*). For $A \subset X$, the w-closure and w-interior of A with respect to w^j denote by $c_{w^j}(A)$ and $i_{w^j}(A)$, respectively, for $j \in \{1,2\}$. For a bi-w space (X, w^1, w^2), a subset A of X is called *closed* if $A = c_{w^1}(c_{w^2}(A))$. The complement of a closed set is called *open*. It is easy to prove that A is closed iff $A = c_{w^1}(A) = c_{w^2}(A)$ iff $A = c_{w^2}(c_{w^1}(A))$. Furthermore, A is open iff $A = i_{w^1}(i_{w^2}(A))$ iff $A = i_{w^1}(A) = i_{w^2}(A)$ iff $A = i_{w^2}(i_{w^1}(A))$.

3 Main Results

Throughout this section, bi-w spaces (X, w^1, w^2) and (Y, ν^1, ν^2) (or simply X and Y) and let $k, l \in \{1,2\}$ be such that $k \neq l$. We now introduce the notion of $w(k,l)$-semi-open set and investigate some of its properties.

Definition 3.1. *A subset A of X is called $w(k,l)$-*semi-open *if $A \subset c_{w^l}(i_{w^k}(A))$. The complement of a $w(k,l)$-semi-open set is called $w(k,l)$-*semi-closed.

Remark 3.2. *A is $w(k,l)$-semi-closed if and only if $X - A$ is $w(k,l)$-semi-open.*

Theorem 3.3. *A is $w(k,l)$-semi-closed if and only if $i_{w^l}(c_{w^k}(A)) \subset A$.*

Proof. Assume that A is $w(k,l)$-semi-closed. Then $X - A$ is $w(k,l)$-semi-open. Thus $X - A \subset c_{w^l}(i_{w^k}(X - A))$, and so $X - A \subset X - i_{w^l}(c_{w^k}(A))$. Then $i_{w^l}(c_{w^k}(A)) \subset A$. Conversely, assume that $i_{w^l}(c_{w^k}(A)) \subset A$. Then $X - A \subset X - i_{w^l}(c_{w^k}(A))$. Thus $X - A \subset c_{w^l}(i_{w^k}(X - A))$. Therefore, $X - A$ is $w(k,l)$-semi-open, and so A is $w(k,l)$-semi-closed. □

Proposition 3.4. *If A_j is $w(k,l)$-semi-open for all $j \in \Lambda$, then $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-semi-open.*

Proof. Assume that A_j is $w(k,l)$-semi-open for all $j \in \Lambda$ and let $A = \bigcup_{j\in\Lambda} A_j$. Fix $j_0 \in \Lambda$. Then $A_{j_0} \subset A$. Thus $c_{w^l}(i_{w^k}(A_{j_0})) \subset c_{w^l}(i_{w^k}(A))$. Since A_{j_0} is $w(k,l)$-semi-open, $A_{j_0} \subset c_{w^l}(i_{w^k}(A))$. Since j_0 is an arbitrary element of Λ, we have $A_j \subset c_{w^l}(i_{w^k}(A))$ for all $j \in \Lambda$. Therefore $\bigcup_{j\in\Lambda} A_j \subset c_{w^l}(i_{w^k}(A))$. Thus $\bigcup_{j\in\Lambda} A_j$ is $w(k,l)$-semi-open. □

Proposition 3.5. *If A_j is $w(k,l)$-semi-closed for all $j \in \Lambda$, then $\bigcap_{j\in\Lambda} A_j$ is $w(k,l)$-semi-closed.*

Proof. The proof is similar to that of Proposition 3.4. □

Theorem 3.6. *A is $w(k,l)$-semi-open if and only if there exists a set U such that $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$.*

Proof. Assume that A is $w(k,l)$-semi-open. Then $A \subset c_{w^l}(i_{w^k}(A))$. Set $U = i_{w^k}(A)$. Then $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$. Conversely, assume that there exists a set U such that $i_{w^k}(U) = U$ and $U \subset A \subset c_{w^l}(U)$. Then $A \subset c_{w^l}(U) = c_{w^l}(i_{w^k}(U)) \subset c_{w^l}(i_{w^k}(A))$. Hence A is $w(k,l)$-semi-open. □

Corollary 3.7. *If there exists $U \in w^k$ such that $U \subset A \subset c_{w^l}(U)$, then A is $w(k,l)$-semi-open.*

Proof. The proof follows from Theorem 3.6. □

Corollary 3.8. *If A is $w(k,l)$-semi-open and $A \subset B \subset c_{w^l}(A)$, then B is $w(k,l)$-semi-open.*

Proof. The proof follows from Theorem 3.6. □

Corollary 3.9. *If A is $w(k,l)$-semi-closed and $i_{w^l}(A) \subset B \subset A$, then B is $w(k,l)$-semi-closed.*

Proof. It follows from Corollary3.8. □

Definition 3.10. *Let (X, w^1, w^2) be a bi-w space and $A \subset X$. The intersection of all $w(k,l)$-semi-closed sets containing A is called the* $w(k,l)$-semi-closure *of A and is denoted by $c_\sigma^{kl}(A)$. The union of all $w(k,l)$-semi-open sets contained in A is called the* $w(k,l)$-semi-interior *of A and is denoted by $i_\sigma^{kl}(A)$.*

Remark 3.11. *Form Definition 3.10, it is obvious that $i_\sigma^{kl}(A) \subset A \subset c_\sigma^{kl}(A)$.*

Lemma 3.12. *$x \in c_\sigma^{kl}(A)$ if and only if $V \cap A \neq \emptyset$ for all $w(k,l)$-semi-open set V containing x.*

Proof. Assume that there exists a $w(k,l)$-semi-open set V containing x such that $V \cap A = \emptyset$. Then $X - V$ is $w(k,l)$-semi-closed such that $x \notin X - V$ and $A \subset X - V$. Hence, $x \notin c_\sigma^{kl}(A)$. Conversely, assume that $x \notin c_\sigma^{kl}(A)$. Then there exists a $w(k,l)$-semi-closed set F such that $A \subset F$ and $x \notin F$. Set $V = X - F$. Thus V is $w(k,l)$-semi-open set containing x such that $V \cap A = \emptyset$. □

Lemma 3.13. *$x \in i_\sigma^{kl}(A)$ if and only if there exists a $w(k,l)$-semi-open set V and x such that $x \in V \subset A$.*

Proof. Assume that $x \in i_\sigma^{kl}(A)$. Then there exists a $w(k,l)$-semi-open set V such that $x \in V \subset A$. Conversely, assume that there exists a $w(k,l)$-semi-open set V such that $x \in V \subset A$. Then $x \in i_\sigma^{kl}(A)$. □

Lemma 3.14. *$i_\sigma^{kl}(X - A) = X - c_\sigma^{kl}(A)$ and $c_\sigma^{kl}(X - A) = X - i_\sigma^{kl}(A)$.*

Proof. Assume that $x \in i_\sigma^{kl}(X - A)$. Then there exists a $w(k,l)$-semi-open set V such that $x \in V \subset X - A$. Thus $V \cap A = \emptyset$, and so $x \notin c_\sigma^{kl}(A)$. Hence $x \in X - c_\sigma^{kl}(A)$. On the other hand, assume that $x \in X - c_\sigma^{kl}(A)$. Then $x \notin c_\sigma^{kl}(A)$, and so there exists a $w(k,l)$-semi-open set V containing x such that $V \cap A = \emptyset$. Then $V \subset X - A$. Hence $x \in i_\sigma^{kl}(X - A)$. This implies that $i_\sigma^{kl}(X - A) = X - c_\sigma^{kl}(A)$. The second part follows from the first part. □

Theorem 3.15. *A is $w(k,l)$-semi-closed if and only if $A = c_\sigma^{kl}(A)$.*

Proof. Assume that A is $w(k,l)$-semi-closed. Then $c_\sigma^{kl}(A) \subset A$. This implies that $A = c_\sigma^{kl}(A)$. Conversely, assume that $A = c_\sigma^{kl}(A)$. By Proposition 3.5, A is $w(k,l)$-semi-closed. □

Theorem 3.16. *A is $w(k,l)$-semi-open if and only if $A = i_\sigma^{kl}(A)$.*

Proof. Assume that A is $w(k,l)$-semi-open. Then $A \subset i_\sigma^{kl}(A)$. This implies that $A = i_\sigma^{kl}(A)$. Conversely, assume that $A = i_\sigma^{kl}(A)$. By Proposition 3.4, A is $w(k,l)$-semi-open. □

Finally, we introduce the concept of the continuity using $w(k,l)$-semi-open sets.

Definition 3.17. *A function $f : X \to Y$ is called* $w(k,l)$-semi-continuous *at x if, for all ν^k-open set V containing $f(x)$, there exists a $w(k,l)$-semi-open set U such that $f(U) \subset V$. We call f* $w(k,l)$-semi-continuous *if f is $w(k,l)$-semi-continuous at x, for all $x \in X$.*

Theorem 3.18. *Let $f : X \to Y$ be a function. The following are equivalent:*

1. *f is $w(k,l)$-semi-continuous.*
2. *$f^{-1}(V)$ is $w(k,l)$-semi-open, for all $V \in \nu^k$.*
3. *$f^{-1}(F)$ is $w(k,l)$-semi-closed, for all ν^k-closed set F.*
4. *$f(c_\sigma^{kl}(A)) \subset c_{\nu^k}(f(A))$, for all $A \subset X$.*
5. *$c_\sigma^{kl}(f^{-1}(B)) \subset f^{-1}(c_{\nu^k}(B))$, for all $B \subset Y$.*
6. *$f^{-1}(i_{\nu^k}(B)) \subset i_\sigma^{kl}(f^{-1}(B))$, for all $B \subset Y$.*

Proof. (1) ⇒ (2): Assume that f is $w(k,l)$-semi-continuous and $V \in \nu^k$. To show that $f^{-1}(V) = i_\sigma^{kl}(V)$, let $x \in f^{-1}(V)$. Then $f(x) \in V$. By assumption, there exists a $w(k,l)$-semi-open set U such that $f(U) \subset V$. Then $x \in U \subset f^{-1}(V)$. Thus $x \in i_\sigma^{kl}(f^{-1}(V))$. This implies that $f^{-1}(V) = i_\sigma^{kl}(V)$. Consequently, $f^{-1}(V)$ is $w(k,l)$-semi-open.

(2) ⇒ (3): Let F be a ν^k-closed set. Then $X - F$ is ν^k-open. By (2), $f^{-1}(X-F)$ is $w(k,l)$-semi-open. Since $f^{-1}(X-F) = X - f^{-1}(F)$, $X - f^{-1}(F)$ is $w(k,l)$-semi-open. Thus $f^{-1}(F)$ is $w(k,l)$-semi-closed.

(3) ⇒ (4): Let $A \subset X$ and $\mathscr{S} = \{F : F \text{ is } \nu^k\text{-closed and } f(A) \subset F\}$. Since $A \subset f^{-1}(f(A))$ and $c_{\nu^k}(f(A)) = \bigcap_{F\in\mathscr{S}} F$, $A \subset f^{-1}(c_{\nu^k}(f(A))) = \bigcap_{F\in\mathscr{S}} f^{-1}(F)$. By (3), $\bigcap_{F\in\mathscr{S}} f^{-1}(F)$ is $w(k,l)$-semi-closed. This implies that $c_\sigma^{kl}(A) \subset \bigcap_{F\in\mathscr{S}} f^{-1}(F) = f^{-1}(c_{\nu^k}(f(A)))$. Thus $f(c_\sigma^{kl}(A)) \subset c_{\nu^k}(f(A))$.

(4) $\Rightarrow$ (5): Let $B \subset Y$. By (4), $f(c_\sigma^{kl}(f^{-1}(B))) \subset c_{\nu^k}(f(f^{-1}(B)))$. This implies that $c_\sigma^{kl}(f^{-1}(B)) \subset f^{-1}(c_{\nu^k}(B))$.

(5) $\Rightarrow$ (6): Let $B \subset Y$. By (5), $c_\sigma^{kl}(f^{-1}(Y-B)) \subset f^{-1}(c_{\nu^k}(Y-B))$. Thus $X - i_\sigma^{kl}(f^{-1}(B)) \subset X - f^{-1}(i_{\nu^k}(B))$. Hence $f^{-1}(i_{\nu^k}(B)) \subset i_\sigma^{kl}(f^{-1}(B))$.

(6) $\Rightarrow$ (1): To show that f is $w(k,l)$-semi-continuous, let $x \in X$ and let V be ν^k-open containing $f(x)$. Thus $V = i_{\nu^k}(V)$. By (6), $f^{-1}(V) = f^{-1}(i_{\nu^k}(V)) \subset i_\sigma^{kl}(f^{-1}(V))$, and so $x \in i_\sigma^{kl}(f^{-1}(V))$. Then there exists a $w(k,l)$-semi-open set G such that $x \in G \subset f^{-1}(V)$. Hence $f(G) \subset V$. Therefore, f is $w(k,l)$-semi-continuous at x. This implies that f is $w(k,l)$-semi-continuous. □

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