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The extension of a linear operator on a-fuzzy normed space when it is fuzzy compact

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Abstract

In this work, our first aim is to prove that every a-fuzzy normed space has a fuzzy complete extension and our other aim is to prove that the extension of a fuzzy compact linear operator defined on a-fuzzy normed space is also a fuzzy compact linear operator.

1 Introduction

In the last few years, Kider [1]-[14] along with his students introduced various types of the notion fuzzy metric space and fuzzy normed space. In 2021, Khudhair and Kider [15], [16], [17], introduced the notion of afuzzy normed space and proved basic important properties of this type of fuzzy normed space. This paper is organized as follows:
In Section 2, we present the a-fuzzy normed space with its important properties. In Section 3, we prove that every a-fuzzy normed space has a fuzzy complete extension and that the extension of a fuzzy compact linear operator defined on a-fuzzy normed space is also a fuzzy compact linear operator.

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2 Basic Concepts

Remark 2.1. *[15] The following are satisfied when* $\circledcirc$ *is a t-conorm:*
(i) $\forall\ 0 < a < 1, 0 < b < 1$, *with* $a > b$, $\exists\ c, 0 < c < 1$ *such that* $a > b \circledcirc c$.
(ii) $\forall\ a, 0 < a < 1, \exists\ 0 < b < 1$ *such that* $b \circledcirc b \leq a$.

Example 2.2. *[14]* $\alpha + \beta - \alpha\beta = \alpha \circledcirc \beta$. *The algebra product is a continuous t-conorm for all* $0 \leq \alpha,\ \beta \leq 1$.

Definition 2.3. *[15] The mapping* $a_{\mathbb{R}} : \mathbb{R} \to I$ *is called a-***fuzzy absolute value on** $\mathbb{R}$ *when* $\circledcirc$ *is a continuous t-conorm if the following are satisfied for all* $\alpha, \beta \in \mathbb{R}$:
(i) $a_{\mathbb{R}}(\alpha) \in (0, 1]$.
(ii) $a_{\mathbb{R}}(\alpha) = 0 \Leftrightarrow if \beta = 0$.
(iii) $a_{\mathbb{R}}(\sigma\alpha) \leq a_{\mathbb{R}}(\sigma).a_{\mathbb{R}}(\alpha)$.
(iv) $a_{\mathbb{R}}(\alpha + \beta) \leq a_{\mathbb{R}}(\alpha) \circledcirc a_{\mathbb{R}}(\beta)$.
Then $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ *is called a-***fuzzy absolute value space**.

Definition 2.4. *[15] If for any arbitrary fuzzy Cauchy sequence* (r_k) *in* $\mathbb{R}$ *is fuzzy,* $\exists\ r \in \mathbb{R}$ *satisfying* $r_k \to r$, *then* $\mathbb{R}$ *is called* **fuzzy complete**.

Definition 2.5. *[15] Suppose* $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ *is a-fuzzy absolute value space and* $n_z : Z \to [0, 1]$ *is a fuzzy set where* $\circledcirc$ *is a continuous t-conorm and* Z *is a vector space over* $\mathbb{R}$. *Then* n_z *is called a-***fuzzy norm on** Z *if it satisfies the following conditions for all* $z,\ w \in Z$ *and for all* $0 \neq \sigma \in \mathbb{R}$:
(i) $n_z(z) \in (0, 1]$.
(ii) $n_z(z) = 0 \Leftrightarrow z = 0$.
(iii) $n_z(\sigma z) \leq a_{\mathbb{R}}(\sigma)\ n_z(z)$.
(iv) $n_z(z + w) \leq n_z(z) \circledcirc n_z(w)$.
Then $(Z, n_z, \circledcirc)$ *is called a-***fuzzy normed space**.

Example 2.6. *[15] Suppose* $\alpha + \beta - \alpha\beta = \alpha \circledcirc \beta$ *for all* $0 \leq \alpha, \beta \leq 1$, $z = C[p, b]$, *and* $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ *is a-fuzzy absolute space define* $n_z(z) = max_{r \in [p,b]} a_{\mathbb{R}}[z(r)]$ *for all* $z \in Z$. *Then* $(Z, n_Z, \circledcirc)$ *is a-fuzzy normed space.*

Lemma 2.7. *[15]* $n_z(z - w) = n_z(w - z)$ *for all* $z, w \in Z$ *when* $(Z, n, \circledcirc)$ *is a-fuzzy normed space.*

Definition 2.8. *[15] Let* $z_k \in Z$ *where* $(Z, n_Z, \circledcirc)$ *is a-fuzzy normed space. Then* (z_k) *is fuzzy convergent to the limit* z *as* $k \to \infty$ *if* $\forall\ s \in (0, 1), \exists\ N \in \mathbb{N} s.t. n_Z(z_k - z) < s\ \forall\ k \geq N$. *If* (z_k) *is fuzzy convergent to the limit* z *or* $\lim_{n\to\infty} n_Z(z_k - z) = 0$. *For simplicity, we write* $\lim_{k\to\infty} z_k = z$ *or* $z_k \to z$ *as* $k \to \infty$.

Definition 2.9. *[15] $fb(w,r) = \{z \in Z : n_z(w-z) < r\}$ is an* **open fuzzy ball** *and $fb[w,r] = \{z \in Z : n_z(w-Z) \leq r\}$ is a* **closed fuzzy ball** *with the center $w \in Z$ and radius r, where $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space.*

Lemma 2.10. *[15] The a-fuzzy norm n_z is a fuzzy continuous function when $(Z, n_Z, \circledcirc)$ and $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ are a-fuzzy normed spaces.*

Definition 2.11. *[15] (z_k) is a fuzzy Cauchy sequence in a-fuzzy normed space $(Z, n_Z, \circledcirc)$, if $\forall\ s \in (0,1), \exists\ N \in \mathbb{N} s.t.\ n_z(z_k - z_m) < s, \forall\ k, m \geq N$.*

Definition 2.12. *[15] If for any arbitrary $w \in W$ and for some $j \in (0,1)$, $fb(w,j) \subseteq W$, then $W \subseteq Z$ is* **fuzzy open** *in the a-fuzzy normed space that $(Z, n_Z, \circledcirc)$. Also, if D^c is fuzzy open, then $D \subseteq Z$ is* **fuzzy closed**. *Moreover, the smallest fuzzy closed set containing H is the* **fuzzy closure** *$\bar{H}$.*

Definition 2.13. *[15] If $\bar{B} = Z$ where $B \subseteq Z$, then B is* **fuzzy dense** *the a-fuzzy normed space $(Z, n_Z, \circledcirc)$.*

Theorem 2.14. *$\bar{H} = Z$ if and only if for any $z \in Z$ there is $w \in H$ with $n_z(z-w) \leq r$ for some $r \in (0,1)$ in a-fuzzy normed space $(Z, n_Z, \circledcirc)$ when $H \subset Z$.*

Definition 2.15. *[15] If for any arbitrary fuzzy Cauchy sequence (z_k) in a-fuzzy normed space $(Z, n_Z, \circledcirc)$ there exists $z \in Z$ with $z_k \to z$, then Z is known as fuzzy complete.*

Theorem 2.16. *[15] $(\mathbb{R}, a_{\mathbb{R}}, \circledcirc)$ is fuzzy complete.*

Theorem 2.17. *[15] If $z_k \to z \in Z$, then (z_k) is fuzzy Cauchy in a-fuzzy normed space $(Z, n_Z, \circledcirc)$.*

Theorem 2.18. *[15] $b \in \bar{B} \Leftrightarrow$ there is $(b_k) \in B$ with $b_k \to b$ when $B \subset Z$ and $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space.*

Theorem 2.19. *[15] If $(Z_1, n_1, \circledcirc)$, $(Z_2, n_2, \circledcirc)$, ... $(Z_k, n_k, \circledcirc)$ are a-fuzzy normed spaces, then $(Z, n, \circledcirc)$ is a-fuzzy normed space where $Z = Z_1 \times Z_2 \times ... \times Z_k$ and $n[(z_1, z_2, ..., z_k)] = n_1(z_1) \circledcirc n_2(z_2) \circledcirc, ..., \circledcirc n_k(z_k)$ for all $(z_1, z_2, ..., z_k) \in Z$.*

Theorem 2.20. *[15] If $(Z_1, n_1, \circledcirc)$, $(Z_2, n_2, \circledcirc)$, ... $(Z_k, n_k, \circledcirc)$ are a-fuzzy normed spaces, then $(Z, n, \circledcirc)$ is fuzzy complete a-fuzzy normed space if and only if $(Z_1, n_1, \circledcirc)$, $(Z_2, n_2$ are fuzzy complete where $Z = Z_1 \times Z_2 \times ... \times Z_k$ and $n[(z_1, z_2, ..., z_k)] = n_1(z_1) \circledcirc n_2(z_2) \circledcirc, ..., \circledcirc n_k(z_k)$ for all $(z_1, z_2, ..., z_k) \in Z$.*

Definition 2.21. *[16] Suppose $(Z, n_z, \circledcirc)$ and $(W, n_w, \circledcirc)$ are two a-fuzzy normed spaces. Then the operator $L : Z \to W$ is called* **fuzzy continuous at** *$z \in Z$. If $\forall\, r \in (0,1)$, $\exists\, t \in (0,1)$, with $n_W[L(z) - L(y)] < r$ for any $y \in Z$ with $n_Z(z - y) < t$. If L is fuzzy continuous $\forall\, z \in Z$, then L is said to be fuzzy continuous on Z.*

Theorem 2.22. *[16] If $B = \{z \in Z : 0 < n_z(z) \leq 1\}$ is fuzzy compact and is a fuzzy closed in Z, then dim $Z < \infty$, where $(Z, n_z, \circledcirc)$ is a-fuzzy normed space.*

Definition 2.23. *[16] The operator $L : D(L) \to Y$ is said to be* **fuzzy bounded** *if there exists $s \in (0,1)$ with $n_Y[L(z)] < s n_z(z)$, for each $z \in D(L)$.............(1)*
where $(Z, n_z, \circledcirc)$ and $(Y, n_Y, \circledcirc)$ are two a-fuzzy normed spaces.

Example 2.24. *[16] Let $Z = C[0,1]$ with $n_z(u) = max_{t\in I} a_{\mathbb{R}}[u(t)]$. Then $(Z, n_z, \circledcirc)$ is a-fuzzy normed space. Let $L : Z \to Z$ be defined by $L[u(t)] = u(t)'$. Then L is linear operator and L is not fuzzy bounded.*

Notation 2.25. *[16] Suppose that $(Z, n_z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are two a-fuzzy normed spaces. Put $afb(Z, W) = \{L : Z \to W\}$, L is a linear fuzzy bounded operator.*

Theorem 2.26. *[16] Define $\mathbf{n_{afb(Z,W)}(l) = sup_{z \in D(L)} n_W(Lz)}$ for all $L \in afb(Z, W)$. Then $[afb(Z, W), n_{afb(Z,W)}, \circledcirc]$ is a-fuzzy normed space when $(Z, n_z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are two a-fuzzy normed spaces.*

Remark 2.27. *[16] Equation (1) can be written as:*
$n_W[L(u)] < n_{afb(Z,W)}(L) n_Z(u)$.

Theorem 2.28. *[16] If W is fuzzy complete then $afb(Z, W)$ is fuzzy complete where $(Z, n_z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are two a-fuzzy normed spaces.*

Theorem 2.29. *[16] Suppose that $(Z, n_z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are two a-fuzzy normed spaces such that $L : D(L) \to W$ is a linear operator where $D(L) \subseteq W$. Then L is fuzzy bounded $\Leftrightarrow$ L is fuzzy continuous.*

Definition 2.30. *[16] An a-fuzzy normed space $(Z, n, \circledcirc)$ is said to be fuzzy compact if $\exists\, \{U_1, U_2, U_3, ..., U_k\} \subseteq \Omega$ such that $Z = \bigcup_{j=1}^{k} U_j$, where Ω is a collection of open fuzzy sets.*

Theorem 2.31. *[16] The a-fuzzy normed space $(Z, n, \circledcirc)$ is fuzzy compact $\Leftrightarrow$ for every arbitrary sequence (z_k) in Z has a subsequence (z_{kj}) with $z_{kj} \to z \in Z$.*

Theorem 2.32. *[16] Let $(Z, n_z, \circledcirc)$ be fuzzy complete a-fuzzy normed space and $W \subseteq Z$. Then W is fuzzy complete if and only if W is fuzzy closed.*

Proposition 2.33. *[16] Z is fuzzy totally bounded if Z is a fuzzy compact when $(Z, n, \circledcirc)$ is a-fuzzy normed space.*

3 The extension of fuzzy compact linear operator

Proposition 3.1. *Let $(Z, n_z, \circledcirc)$ be a-fuzzy normed space with (z_k) and (v_k) being two fuzzy Cauchy sequences in Z. Then*
(1) $(z_k + v_k)$ is a fuzzy Cauchy sequence in Z.
(2) (cz_k) is a fuzzy Cauchy sequence in Z for every $0 \neq c \in \mathbb{R}$.

Proof. (1) Since (z_k) and (v_k) are two fuzzy Cauchy sequences in Z, for any $t, r \in (0, 1)$ there is N_1, N_2 such that $n_Z(z_j - z_m) < t$ for all $j, m > N_1$ and $n_Z(v_j - v_m) < r$ for all $j, m > N_2$. Now choose $N = N_1 \vee N_2$. Then $\forall\ j, m > N$, $n_Z[(z_j + v_j) - (z_m + v_m)] \leq n_Z(z_j - z_m) \circledcirc n_Z(v_j - v_m) < t \circledcirc r$. Let $q \in (0, 1)$ be chosen such that $t \circledcirc r < q$. This implies that $n_U[(z_j + v_j) - (z_m + v_m)] < q$, $\forall\ j, m > N$. Hence $(z_k + v_k)$ is a fuzzy Cauchy sequences in Z.
(2) Since (z_k) is fuzzy Cauchy sequences in Z, $\forall\ t \in (0, 1)$, $\exists\ N$ with $n_Z(z_j - z_m) < t$, $\forall\ j, m > N$. Now for every $0 \neq c \in \mathbb{R}$, $n_Z(cz_j - cz_m) \leq a_{\mathbb{R}}(c).t$. Let $p \in (0, 1)$ be chosen such that $a_{\mathbb{R}}(c).t < p$. This implies that $n_Z(cz_j - cz_m) < p$, $\forall\ j, m > N$. Hence (cz_k) is a fuzzy Cauchy sequences in Z for every $0 \neq c \in \mathbb{R}$. □

Lemma 3.2. *For any a-fuzzy normed space $(U, n_U, \circledcirc)$, there exists a-fuzzy normed space $(\tilde{U}, \tilde{n}, \circledcirc)$.*

Proof. Let (u_k) and (u'_k) be two fuzzy Cauchy sequences in U. We say that (u_k) is equivalent to (u'_k), written as $(u_k) \sim (u'_k)$, if $\lim_{k\to\infty} n_U(u_k - u'_k) = 0$. Let $\tilde{U}$ be the set of all equivalence classes $\tilde{u}, \tilde{v}, \tilde{w},$ where $\tilde{u} = [(u_k)] = \{(u'_k))$ is a fuzzy Cauchy sequence in U and $(u'_k) \sim (u_k)\}$. We wish to make $\tilde{U}$ a vector space in order to define on $\tilde{U}$ the two algebraic operations of a vector space. Consider any two elements $\tilde{u}, \tilde{v}$ in $\tilde{U}$, and any representation $(u_k) \in \tilde{u}, (v_k) \in \tilde{v}$. We set $z_k = u_k + v_k$. Then (z_k) is a fuzzy Cauchy sequence by Proposition 3.1. Thus we define $\tilde{z} = \tilde{u} + \tilde{v}$ where $\tilde{z}$ is the class for which (z_k) is a representative. Now we show that this definition is independent of particular choice of a fuzzy Cauchy sequence belonging to $\tilde{u}$ and $\tilde{v}$. In fact, if $(u'_k) \sim (u_k)$ and $(v'_k) \sim (v_k)$, then $(u'_k + (v'_k)) \sim (u_k + (v_k))$ because

$n_U[(u_k+v_k)-(u'_k+v'_k)] \le n_U(u_k-u'_k) \circledcirc n_U(v_k-v'_k)$. Similarly, we define $c\tilde{u} \in \tilde{U}$ to be the class for which cu_k is a representative. The zero element $\tilde{O}$ is the equivalence class containing all Cauchy sequences which fuzzy converge to zero. It is not difficult to show that the two algebraic operations have all the properties required for $\tilde{U}$ to be a vector space. Now define
$\tilde{n}(\tilde{u})= \lim_{k\to\infty} n_U(u_k)$ ....(2)
where $(u_k) \in \tilde{u}$. First, we show that this limit exists.
$n_U(u_k) = n_U(u_k - u_m + u_m) \le n_U(u_k - u_m) \circledcirc n_U(u_m) = n_U(u_k - u_m) + n_U(u_m) - n_U(u_k - u_m).n_U(u_m)$.
$n_U(u_k) - n_U(u_m) \le n_U(u_k - u_m) - n_U(u_k - u_m).n_U(u_m)$ ....(3).
$a_{\mathbb{R}}[n_U(u_k) - n_U(u_m)] \le n_U(u_k - u_m) - n_U(u_k - u_m).n_U(u_m)$. Since (u_k) is a fuzzy Cauchy sequence, so we can choose $t \in (0,1)$ so that $a_{\mathbb{R}}[n_U(u_k) - n_U(u_m)] < t$. This implies that the limit in (2) exists since $(R, a_R, \circledcirc)$ is fuzzy complete by Theorem 2.16. Next, we show that the limit in (1) does not depend on the particular choice of the representative of the classes. If $(u'_k) \sim (u_k)$, then $n_U(u_k - u'_k) \to 0$ as $k \in \infty$. But $n_U(u_k - u'_k) \le n_U(u_k) \circledcirc n_U(u'_k)$. So $[\lim_{k\to\infty} n_U(u_k) \circledcirc \lim_{k\to\infty} n_U(u'_k)] = 0$. Therefore, $\lim_{k\to\infty} n_U(u_k) = \lim_{k\to\infty} n_U(u'_k)$.
Now we show that $(\tilde{U}, \tilde{n}, \circledcirc)$ is a-fuzzy normed space.
(i) Since $0 < n_U(u_k) \le 1$, $0 < \tilde{n}(\tilde{u}) \le 1$ for all $\tilde{u} \in \tilde{U}$.
(ii) $\tilde{n}(\tilde{u}) = 0 \Leftrightarrow \lim_{k\to\infty} n_U(u_k) = 0 \Leftrightarrow u_k = 0 \Leftrightarrow \tilde{u} = \tilde{0}$.
(iii) $\tilde{n}(c\tilde{u}) = \lim_{k\to\infty} n_U(cu_k) = \lim_{k\to\infty} a_{\mathbb{R}}(c) n_U(u_k) = a_{\mathbb{R}}(c) \lim_{k\to\infty} n_U(u_k) = a_{\mathbb{R}}(c)\tilde{n}(\tilde{u})$ for every $0 \ne c \in \mathbb{R}$.
(iv) To prove $\tilde{n}(\tilde{u} + \tilde{v}) \le \tilde{n}(\tilde{u} \circledcirc \tilde{n}(\tilde{v})$, $(u_k) \in \tilde{u}, (v_k) \in \tilde{v}$. We know that, $n_U(u_k + v_k) \le n_U(u_k) \circledcirc n_U(v_k)$. $\lim_{k\to\infty} n_U(u_k + v_k) \le \lim_{k\to\infty} n_U(u_k) \circledcirc \lim_{k\to\infty} n_U(v_k)$. We have, $\tilde{n}(\tilde{u} + \tilde{v}) \le \tilde{n}(\tilde{u} \circledcirc \tilde{n}(\tilde{v})$.
This implies that $(\tilde{U}, \tilde{u}, \circledcirc)$ is a-fuzzy normed space. □

Lemma 3.3. *If $(U, n_U, \circledcirc)$ is an a-fuzzy normed space, then U is fuzzy isomorphic to a subspace $\tilde{W}$ of $\tilde{U}$ and $\tilde{W}$ is fuzzy dense in $\tilde{U}$.*

Proof. From Lemma 3.2, we have $(\tilde{U}, \tilde{n}, \circledcirc)$ is a-fuzzy normed space. Now, for any $\tilde{w} \in \tilde{U}$, let $\tilde{w}$ be the class of constant fuzzy Cauchy sequence $(w, w, w, ..., w, ...)$ and let $\tilde{W}$ be a subspace of $\tilde{U}$ of such classes. Define the operator $S : U \to \tilde{W}$ by $S(w) = \tilde{w}$ where $(w, w, w, ..., w, ...) \in \tilde{w}$ and the subspace $\tilde{W} = S(U)$. Since $\tilde{n}(\tilde{w}) = w$, S is fuzzy isometric and it is well known that every fuzzy isometry is injective but S is surjective since $\tilde{w} = S(U)$. Hence $\tilde{w} \cong S(U)$. Now we prove that $\tilde{W}$ is fuzzy dense in $\tilde{U}$. Consider $\tilde{u} \in \tilde{U}$ and let $(u_k) \in \tilde{u}$ so $\forall\, t, 0 < t < 1\ \exists\, N$ with $n_U(u_k - u_N) < t$ for all $k > N$. Let $(u_N, u_N, ..., u_N, ...) \in \check{u}_N$. Then $\tilde{n}(\tilde{u} - \check{u}_N) = \lim_{k\to\infty} n_U(u_k - u_N) < t$. Hence $\tilde{W}$ is fuzzy dense in $\tilde{U}$ by Theorem 2.14. □

Lemma 3.4. *The space $(\tilde{U}, \tilde{n}, \circledcirc)$ is fuzzy complete, if $(U, n_U, \circledcirc)$ is an a-fuzzy normed space.*

Proof. Let $(\breve{u}_k)$ be a fuzzy Cauchy sequence in $\tilde{U}$. Using the fuzzy denseness of $\tilde{W}$, we have for each $\breve{u}_k \in \tilde{U}$ there exists $\breve{z}_k \in \tilde{W}$ such that $\tilde{n}(\breve{u}_k - \breve{z}_k) < \frac{1}{k}$. Hence $\tilde{n}(\breve{z}_j - \breve{z}_k) \leq \tilde{n}(\breve{z}_j - \breve{u}_j) \circledcirc \tilde{n}(\breve{u}_j - \breve{u}_k) \circledcirc \tilde{n}(\breve{u}_k - \breve{z}_j) \leq \frac{1}{j} \circledcirc \tilde{n}(\breve{u}_j - \breve{u}_k) \circledcirc \frac{1}{k}$ let $t \in (0,1)$ such that $\frac{1}{j} \circledcirc \tilde{n}(\breve{z}_j - \breve{u}_j)\frac{1}{k}t$; that is, $\tilde{n}(\tilde{z}_j - \tilde{z}_k) < t$. Hence $(\breve{z}_k)$ is a fuzzy Cauchy sequence but $S : U \to \tilde{W}$ is fuzzy isometric by Lemma 3.3 with $\breve{z}_j \in \tilde{W}$. So the sequence $(z_j) = (S^{-1}(\breve{z}_j))$ is a fuzzy Cauchy sequence in U. Let $\tilde{u}$ be the class for which (z_j) belongs. Now we prove that $\breve{u}_k \to \tilde{u}$. $\tilde{n}(\breve{u}_k - \tilde{u}) \leq \tilde{n}(\breve{u}_k - \breve{u}_k) \circledcirc \tilde{n}(\breve{z}_k - \tilde{u})$. But $(z_j) \in \tilde{u}$ and $\breve{z}_k \in \tilde{W}$. As a result, $(z_k, z_k, ..., z_k, ...) \in \breve{z}_k$. Thus $\tilde{n}(\breve{u}_k - \tilde{u}) \leq \frac{1}{k} \circledcirc \lim_{m\to\infty} n_U(z_n - z_m)$. Let $r \in (0,1)$ be chosen such that $\frac{1}{k} \circledcirc \lim_{m\to\infty} n_U(z_n - z_m) < r$. This implies that $\tilde{n}(\breve{u}_k - \tilde{u}) < r$, that is, $\breve{u}_k \to \tilde{u}$. Hence $(\tilde{U}, \tilde{n}, \circledcirc)$ is fuzzy complete. □

Lemma 3.5. *The space $(\tilde{U}, \tilde{n}, \circledcirc)$ is unique except for isometries, if $(U, n_U, \circledcirc$ is an a-fuzzy normed space.*

Proof. Let $\hat{W}$ be a subspace of $\hat{U}$ which is fuzzy dense in $\hat{U}$ and where $(\hat{U}, \hat{n}, \circledcirc)$ is another a-fuzzy normed space which is fuzzy isometric with U through the operator T; that is, $U = T(\hat{W})$. Since $\hat{W} \cong U$ and $\tilde{W} \cong U$, $\hat{W} \cong \tilde{W}$. Also if $\hat{y}_k \to \hat{y}$, then $\hat{n}(\hat{y}_k - \hat{y}) \to 0$ as $k \to \infty$. Consequently, $\hat{n}(\hat{y}) = \lim_{k\to\infty} n_U(y_k)$. Thus the a-fuzzy norm on $\hat{U}$ and $\tilde{U}$ must be the same. Hence $\hat{U} \cong \tilde{U}$. □

Theorem 3.6. *If $(U, n_U, \circledcirc)$ is an a-fuzzy normed space, then there exists a fuzzy complete a-fuzzy normed space $(\tilde{U}, n_{\tilde{U}}, \circledcirc)$ having a subspace $\tilde{W}$ which is fuzzy isometric to U and fuzzy dense in $\tilde{U}$. Also, the space $\tilde{U}$ is unique except for fuzzy isometries.*

Proof. By Lemma 3.2, the a-fuzzy normed space $(\tilde{U}, n_{\tilde{U}}, \circledcirc)$ is constructed and by Lemma 3.3 a fuzzy isometric S from U onto $\tilde{W}$ is defined and $\tilde{W}$ is fuzzy dense in $\tilde{W}$ follows. By Lemma 3.4, the space $\tilde{U}$ is fuzzy complete. Finally, by Lemma 3.5, uniqueness of $\tilde{U}$ except for isometries follows. □

Definition 3.7. *Let W be a subset of a-fuzzy normed space $(Z, n_Z, \circledcirc)$ and let $\alpha \in (0,1)$. A set $B_\alpha \subseteq W$ is an α-fuzzy net for W if for every $z \in W$ there is $b \in B_\alpha$ such that $n_Z(z-b) < \alpha$. The set W is called* **totally fuzzy bounded** *if for every $\alpha \in (0,1)$ there is a finite α-fuzzy net $B_\alpha = \{b_1, b_2, , b_k\} \subseteq W$. In other words, totally fuzzy boundedness of W means that for every given $\alpha \in (0,1)$ the set W is contained in the union of finitely many open fuzzy balls of radius α.*

Theorem 3.8. *W is totally fuzzy bounded, if W is relatively fuzzy compact when $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space with $W \subseteq Z$.*

Proof. Suppose that W is relatively fuzzy compact. We prove that for any fixed given $\alpha \in (0,1)$ there exists a finite α-fuzzy net for W. If $W = \varnothing$, then $\varnothing$ is an α-fuzzy net for W. Let $W \neq \varnothing$. We choose $b_1 \in W$ if $n_Z(b_1 - z) < \alpha$ for all $z \in W$. Then $\{b_1\}$ is an α-fuzzy net for W. Otherwise, let $b_2 \in W$ be such that $n_Z(b_1 - b_2) \geq \alpha$. If for all $z \in W$, $n_Z(b_2 - z) < \alpha$ and $n_Z(b_1 - z) < \alpha$. Otherwise, let $b_3 \in W$ be such that $n_Z(b_1 - b_3) \geq \alpha$, $n_Z(b_2 - b_3) \geq \alpha$. If for all $z \in W$ $n_Z(b_j - z) < \alpha$ for $j = 1, 2, 3$, then $\{b_1, b_2, b_3\}$ is an α-fuzzy net for W. Continue this process by selecting $b_4 \in W$, ,etc. We assert the existence of positive integer k such that the set $\{b_1, b_2, b_3, , b_k\}$ obtained after k such steps is an α-fuzzy net for W. In fact, if there were no such k, our construction would give a sequence (b_j) satisfying $n_Z(b_j - b_n) \geq \alpha$ for all $j \neq n$. Obviously, (b_j) could not have a subsequence which is fuzzy Cauchy. Hence (b_j) could not have a subsequence which is fuzzy convergent in W. This contradicts the relative compactness of W because (b_j) lies in W by the construction. Consequently, there must be a finite α-fuzzy net for W. Since $\alpha \in (0,1)$ was arbitrary, W is totally fuzzy bounded. □

Theorem 3.9. *W is relatively fuzzy compact, if W is totally fuzzy bounded and Z is fuzzy complete when $(Z, n_Z, \circledcirc)$ is a-fuzzy metric space with $W \subseteq Z$.*

Proof. Suppose that W is totally fuzzy bounded and Z is fuzzy complete. Consider any sequence (b_j) in W. We prove that (b_j) has a subsequence which fuzzy converge in W so that W will be then relatively fuzzy compact. Now, by assumption, W has a finite α-fuzzy net. By Definition 3.7, W is contained in the union of finitely many open fuzzy balls of radius $\alpha = 1$. From these fuzzy balls we can choose a fuzzy ball B_1 which contains infinitely many terms of (b_j). Let $(b_j^{(1)})$ be a subsequence of (b_j) which lies in B_1. Similarly, by assumption, W is also contained in the union of finitely many fuzzy balls of radius $\alpha = \frac{1}{2}$. From these fuzzy balls we can choose a fuzzy ball B_2 which contains a subsequence $(b_j^{(2)})$ of the subsequence $(b_j^{(1)})$. Continue by induction choosing $\alpha = \frac{1}{3}$, $\alpha = \frac{1}{4}$, .. and setting $y_j = b_j^{(j)}$. Then for every given $\alpha \in (0,1)$ there is N such that all y_j with $j > N$ lie in the fuzzy ball of radius α. Hence (y_j) is a fuzzy Cauchy sequence and so it is fuzzy convergent in Z; say, $y_j \to y \in Z$ since Z is fuzzy complete. Also $y_j \in W$ implies $y \in W$. By the definition of fuzzy closure for every sequence (z_j) in $\bar{W}$, there is a sequence (x_j) in W which satisfies $n_Z(z_j - x_j) \leq \frac{1}{j}$ for every j. Since (x_j) in W, it has a subsequence which is fuzzy convergent in $\bar{W}$

as we have just shown. Hence (z_j) also has a subsequence which is fuzzy convergent in $\bar{W}$ since, $n_Z(z_j - x_j) \leq \frac{1}{j}$ so that $\bar{W}$ is fuzzy compact and W is relatively fuzzy compact. □

Theorem 3.10. *If Z is totally fuzzy bounded, then for every $\alpha \in (0,1)$, W has a finite α-fuzzy net $B_\alpha \subseteq W$; that is, W is totally fuzzy bounded when $(Z, n_Z, \circledcirc)$ is an a-fuzzy metric space and $W \subseteq Z$.*

Proof. The case $W = \varnothing$ is obvious. Assume that $W \neq \varnothing$ by assumption an $\in (0,1)$ be given, there is a finite α_1-fuzzy net $B_{\alpha_1} \subseteq Z$ for W where $\alpha_1 = \frac{\alpha}{2}$. Hence W is contained in the union of finitely fuzzy balls of radius α_1 with the elements of B_{α_1} as centers. Let $B_1, B_2, ..., B_k$ be those fuzzy balls which intersect W and let $b_1, b_2, ..., b_k$ be their centers. We pick a point $z_j \in (W \cap B_j)$. Then $B_\alpha = \{z_1, z_2, , z_k\} \subseteq W$ is an α-fuzzy net for W because for every $z \in W$ there is a B_j containing z and $n_Z(z - z_j) \leq n_Z(z - b_j) \circledcirc n_Z(b_j - z_j) \leq \alpha_1 \circledcirc \alpha_1 < \alpha$. □

Definition 3.11. *The a-fuzzy normed space $(Z, n_Z, \circledcirc)$ is said to be fuzzy separable if it contains a countable fuzzy dense subset.*

Theorem 3.12. *If W is totally fuzzy bounded, then W is separable when $(Z, n_Z, \circledcirc)$ is a-fuzzy normed space and $W \subseteq Z$.*

Proof. If W is totally fuzzy bounded, then by Lemma 2.33 the set W contains a finite α-fuzzy net B_α for itself where $\alpha = \alpha_k = \frac{1}{k}$, $k = 1, 2, ...$ Then $B = \cup_{k=1}^{\infty} B_{\alpha_k}$ is countable and B is fuzzy dense in W. In fact, for any given $\alpha \in (0,1)$ there is k such that $\frac{1}{k} < \alpha$. Hence for any $z \in W$ there is an $a \in B_{\frac{1}{k}} \subseteq B$ such that $m(z, a) < \alpha$. This shows that W is separable. □

Theorem 3.13. *If S is fuzzy compact, then the range $R(S)$ is fuzzy separable when $(Z, n_Z, \circledcirc)$ and $(W, n_W, \circledcirc)$ are a-fuzzy normed spaces and $S : Z \to W$ is a linear operator.*

Proof. Consider the fuzzy open balls $B_k = fb(b, \frac{1}{k}) \subseteq Z$. Since S is fuzzy compact the image $C_k = (B_k)$ is relatively compact and C_k are separable by Theorem 3.11. The a-fuzzy norm of any $z \in Z$ is $n_Z(z) < \frac{1}{k}$. Hence $z \in B_k$. Consequently,
$B = \cup_{k=1}^{\infty} B_k$ ....(a)
and
$S(B) = \cup_{k=1}^{\infty} S(B_k) = \cup_{k=1}^{\infty} C_k$ ....(b)
Since C_k are separable, it has a countable fuzzy dense subset D_k and $D = \cup_{k=1}^{\infty} D_k$ is countable.
Equation (b) shows that D is fuzzy dense in the range $R(S) = S(Z)$. □

Definition 3.14. *A is called* **relatively fuzzy compact** *if* $\bar{A}$ *is fuzzy compact where* $(Z, n_Z, \circledcirc)$ *is an a-fuzzy normed space and* $A \subseteq Z$.

Definition 3.15. *The linear operator* $S : Z \to W$ *is called* **fuzzy compact operator** *if S maps every fuzzy bounded subset A of Z to a relatively fuzzy compact S(A) of W where* $(Z, n_Z, \circledcirc)$ *and* $4(W, n_W, \circledcirc)$ *are a-fuzzy normed spaces.*

Theorem 3.16. *Every fuzzy compact linear operator* $S : Z \to W$ *is fuzzy bounded and hence fuzzy continuous when* $(Z, n_Z, \circledcirc)$ *and* $4(W, n_W, \circledcirc)$ *are a-fuzzy normed spaces.*

Proof. The fuzzy ball $B = \{z \in Z : n_Z(z) = 1\}$ is fuzzy bounded since S is fuzzy compact, $S(\bar{B})$ is fuzzy compact and is fuzzy bounded by Proposition 2.33 so that $sup_{z\in B} n_W[S(z)] < \infty$. Hence S is fuzzy bounded and, by Theorem 2.29, it is fuzzy continuous. □

Theorem 3.17. *The sequence* $(S(z_k))$ *has a fuzzy convergent subsequence for every fuzzy bounded sequence* (z_k) *in Z if and only if the operator S is fuzzy compact where* $(Z, n_Z, \circledcirc)$ *and* $(W, n_W, \circledcirc)$ *are a-fuzzy normed spaces and* $S : Z \to W$ *is a linear operator.*

Proof. If S is fuzzy compact and the sequence (z_k) is fuzzy bounded in Z, the closure of $(S(z_k))$ is fuzzy compact in W and definition 3.14 shows that $(S(z_k))$ has a convergent subsequence.
For the converse, suppose that every fuzzy bounded sequence (z_k) in Z contains a subsequence (z_{k_j}) such that $(S(z_{k_j}))$ is fuzzy convergent in W. Consider any fuzzy bounded subset B of Z and let (w_k) be any sequence in $S(B)$. Then $S(z_k) = w_k$ for some $z_k \in B$ and (z_k) is fuzzy bounded since B is fuzzy bounded. By our assumption, $(S(z_k))$ contains a fuzzy convergent subsequence. Hence $(S(B))$ is fuzzy compact by definition 3.14 because (w_k) was arbitrary in $S(B)$. Hence S is fuzzy compact. □

Theorem 3.18. *Let* $(Z, n_Z, \circledcirc)$ *and* $(W, n_W, \circledcirc)$ *be a-fuzzy normed spaces where W is fuzzy complete and* $S : Z \to W$ *is a linear operator. If S is fuzzy compact, then it has a linear fuzzy compact extension* $\tilde{S} : \tilde{Z} \to W$ *where* $\tilde{Z}$ *is the fuzzy completion Z.*

Proof. We may regard Z as a subspace of $\tilde{Z}$ since S is fuzzy bounded by Theorem 3.16 and so it has a fuzzy bounded extension $\tilde{S} : \tilde{Z} \to W$. We prove that fuzzy compactness of S implies that $\tilde{S}$ is also fuzzy compact. Consider any sequence $(\widetilde{z}_k)$ in Z. We show that $S(\widetilde{z}_k)$ has a fuzzy convergent subsequence. Since Z is fuzzy dense in $\tilde{Z}$, there is a sequence (z_k) in Z such that $\widetilde{z_k} - z_k \to 0$. It is clear that (z_k) is fuzzy bounded. Now, using the fuzzy compactness S, we have $S(z_k)$ has a fuzzy convergent subsequence $S(z_{k_j})$. Let $S(z_{k_j}) \to w_0 \in W$. Now $\widetilde{z_k} - z_k \to 0$ implies $\widetilde{z_{k_j}} - z_{k_j} \to 0$ since $\tilde{S}$ is fuzzy bounded so it is fuzzy continuous by Theorem 2.29. Thus we obtain $\tilde{S}\widetilde{z_{k_j}} - Sz_{k_j} = \tilde{S}[\widetilde{z_k} - z_k] \to \tilde{S}(0) = 0$. By (3), this implies that $\tilde{S}\widetilde{z_{k_j}} \to w_0$. We have shown that the arbitrary sequence $\widetilde{z_k}$ has a subsequence $\widetilde{z_{k_j}}$ such that $\tilde{S}\widetilde{z_{k_j}}$ is fuzzy convergent. This proves fuzzy compactness of S by Theorem 3.17. □

4 Conclusion

In this paper, we first showed that every a-fuzzy normed space has a fuzzy complete extension. Then we used the definition of a fuzzy compact linear operator to prove that the extension of a fuzzy compact linear operator on an a-fuzzy normed space is again a fuzzy compact linear operator. For future work, we recommended discussing the spectral properties of fuzzy compact linear operators defined on a-fuzzy normed spaces.

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