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The Improved 6^{th} Order Runge-Kutta Method for Solving Initial Value Problems

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Abstract

To numerically solve initial value problems (IVPs), in this paper we construct and apply an improved 6^{th} order Runge-Kutta method

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(IRK6M) which is based on the traditional Runge-Kutta method (RMK), but with two steps. The method produces results that were close to the 6^{th} order Runge-Kutta method (RK6M), but with fewer numbers of functions evaluations. We also determine the order conditions of the method. To show the method's effectiveness, an illustrative problem is solved. We compare our results to the exact solution as well as the (RK6M), so it is supported by Table and Figure using MATLAB.

1 Introduction

Differential equations appear in a variety of physical, chemical, biological, and engineering phenomena. Moreover, differential equations represent the majority of the laws that describe these phenomena. Many researchers have been interested in solving IVPs to obtain approximate solutions using different numerical methods. Some authors have improved the efficiency of RKM by increasing the Taylor series terms. Other researchers are currently working to improve the Runge-Kutta method for the fourth and fifth orders, but with a fewer number of functions evaluation, as shown in [1-5]. The main aim of this paper is to improve the RK6M to obtain the order conditions. Then we use an illustrative example to examine the method effectively.

2 Derivation of the method

Consider the traditional general form of IVP of the form

$$u' = \phi(t, u(t)), u(t_0) = u_0, t \in [0, 1] \tag{2.1}$$

To get an approximate solution of $u(t)$, the approach of the IRK6M is proceed to compute u_{n+1} as an approximation to $u(t_n + h)$.

$$u_{n+1} = u_n + h\Big(b_1 K_1 + b_{-1} K_{-1} + \sum_{i=2}^{7} b_i (K_i - K_{-i})\Big) \tag{2.2}$$

where

$$K_1 = \phi(t_n, u(t_n)) \tag{2.3}$$

$$K_{-1} = \phi(t_{n-1}, u(t_{n-1})) \tag{2.4}$$

$$K_i = \phi(t_n + c_i h, u_n + h \sum_{j=1}^{i-1} a_{ij} K_j) \tag{2.5}$$

$$K_{-i} = \phi(t_n + c_i h, u_{n-1} + h\sum_{j=1}^{i-1} a_{ij}K_{-j}) \tag{2.6}$$

For $c_i \in [0,1]$, $i = 2, ..., 7$ and satisfying the condition $c_i = \sum_{j=1}^{i-1} a_{ij}$, to determine the coefficients, we expand 2.5,2.6 using Taylor's series expansion and the results equated with $u(t_n + h)$ given by

$$u(t_n + h) = u(t_n) + hu'(t_n) + \frac{h^2}{2!}u''(t_n) + \cdots + \frac{h^6}{6!}u^{(6)}(t_n) + o(h^7) \tag{2.7}$$

First, we calculate the successive derivatives of the equation 2.1 up to the sixth derivative as follows

$$u''(t) = \phi_x + \phi_u\phi \tag{2.8}$$

$$u'''(t) = (\phi_{x^2} + 2\phi_{ux}\phi + \phi_{u^2}\phi^2) + \phi_u(\phi_x + \phi_u\phi) \tag{2.9}$$

$$\begin{aligned} u^{(4)}(t) = (\phi_{x^3} + 3\phi_{x^2u}\phi + 3\phi_{xu^2}\phi^2 + \phi_{u^3}\phi^3) + \phi_u(\phi_{x^2} + 2\phi_{ux}\phi + \phi_{u^2}\phi^2) \\ + (3\phi_{u^2}\phi + 3\phi_{xu} + \phi_{u^2})(\phi_x + \phi_u\phi) \end{aligned} \tag{2.10}$$

$$\begin{aligned} u^{(5)}(t) = (\phi_{x^4} + 4\phi_{x^3u}\phi + 6\phi_{x^2u^2}\phi^2 + 4\phi_{xu^3}\phi^3 + \phi_{u^4}\phi^4) + \phi_u(\phi_{x^3} + 3\phi_{x^2u}\phi + \\ 3\phi_{xu^2}\phi^2 + \phi_{u^3}\phi^3) + (3\phi_{u^2}\phi + 3\phi_{xu} + \phi_{u^2})(\phi_x + \phi_u\phi) \end{aligned} \tag{2.11}$$

$$\begin{aligned} u^{(6)}(t) = (\phi_{x^5} + 5\phi_{x^4u}\phi + 10\phi_{x^3u^2}\phi^2 + 10\phi_{x^2u^3}\phi^3 + 5\phi_{xu^4}\phi^4 + \phi_{u^5}\phi^5) + \phi_u(\phi_{x^4} + \\ 4\phi_{x^3u}\phi + 6\phi_{x^2u^2}\phi^2 + 4\phi_{xu^3}\phi^3 + \phi_{u^4}\phi^4) + (3\phi_{u^2}\phi + 3\phi_{xu} + \cdots + \phi_u\phi) \end{aligned} \tag{2.12}$$

By plugging 2.8-2.12 into 2.7, we get

$$\begin{aligned} u_{n+1} = u_n + h\phi + \frac{h^2}{2!}(\phi_x + \phi_u\phi) + \frac{h^3}{3!}\big((\phi_{x^2} + 2\phi_{ux}\phi + \phi_{u^2}\phi^2) + \phi_u(\phi_x + \phi_u\phi)\big) + \ldots \\ + \frac{h^6}{6!}u^{(6)}\big(\phi_{x^5} + 5\phi_{x^4u}\phi + 10\phi_{x^3u^2}\phi^2 + 10\phi_{x^2u^3}\phi^3 + 5\phi_{xu^4}\phi^4 + \phi_{u^5}\phi^5) + \phi_u(\phi_{x^4} + \\ 4\phi_{x^3u}\phi + 6\phi_{x^2u^2}\phi^2 + 4\phi_{xu^3}\phi^3 + \phi_{u^4}\phi^4) + (3\phi_{u^2}\phi + 3\phi_{xu} + \cdots + o(h^7) \end{aligned} \tag{2.13}$$

Define $F_n = \sum_{k=0}^{n} \binom{n}{k} \phi_{x^{n-k}u^k}\phi^k$, for $n = 1, \ldots, 5$, 2.13 can be written as

$$\begin{aligned} u_{n+1} - u_n = h\phi + \frac{h^2}{2!}F_1 + \frac{h^3}{3!}\big(F_2 + \phi_u F_1\big) + \frac{h^4}{4!}\big(F_3 + \phi_u F_2 + (3\phi_{u^2}\phi + 3\phi_{xu} + \phi_{u^2}) \\ F_1\big) + \frac{h^5}{5!}\big(F_4 + \phi_u F_3 + (3\phi_{u^2}\phi + \ldots\big) + \frac{h^6}{6!}\big(F_5 + \cdots + o(h^7) \end{aligned} \tag{2.14}$$

After some algebraic simplification, we get a system of nonlinear equations whose solution represents the so-called order conditions

3 Order conditions

Using the Taylor series expansion up to order six for $k_1, k_2, ..., k_7$ and $k_{-1}, ...,$ k_{-7} which were used in equations 2.3-2.6, we have

$$K_1 = \phi \tag{3.15}$$

$$\begin{aligned} k_{-1} = \phi + \frac{h}{2}F_1 + \frac{h^2}{6}\big(F_2 + \phi_u F_1\big) + \frac{h^3}{24}\big(F_3 + \phi_u F_2 + (3\phi_{u^2}\phi + 3\phi_{xu} + \phi_{u^2})F_1\big) \\ + \frac{h^4}{120}\big(F_4 + \phi_u F_3 + (3\phi_{u^2}\phi + 3\phi_{xu} + \phi_{u^2})F_1\big) + \cdots + o(h^7) \end{aligned} \tag{3.16}$$

$$\begin{aligned} k_7 = \phi + \frac{h}{2}c_7F_1 + \frac{h^2}{6}\big(c_7^3F_2 + 2c_7a_{21}\phi_uF_1\big) + \frac{h^3}{24}\big(c_7^3F_3 + 2c_7^2a_{21}^2\phi_uF_2 + (3\phi_{u^2}\phi \\ + 3\phi_{xu} + \phi_{u^2})F_1\big) + \frac{h^4}{120}\big(c_7^4F_4 + 2c_7^3a_{21}^3\phi_uF_3 + (3\phi_{u^2}\phi + \cdots + o(h^7)\big)\big) \end{aligned} \tag{3.17}$$

$$\begin{aligned} k_{-7} = \phi+\frac{h}{2}(1-c_7)F_1+\frac{h^2}{6}\big((1-c_7)^2F_2+(1-2c_7a_{21})^2\phi_uF_1\big)+\frac{h^3}{24}\big(1-c_7)^3F_3+(1 \\ - 2c_7a_{21})^3\phi_uF_2 + (3\phi_{u^2}\phi + 3\phi_{xu} + \phi_{u^2})F_1\big) + \cdots + o(h^7) \end{aligned} \tag{3.18}$$

Substituting the above formulas 3.15-3.18 into 2.2, we get.

$$\begin{aligned} u_{n+1} = u_n+h(b_1-b_{-1})F_1+h^2(b_{-1}+b_2+b_3+\cdots+b_7)+\frac{h^3}{2}(b_{-1}+(1-2c_2)b_2+\dots \\ -(1-2c_7)b_7)(F_2 + \phi_uF_1) + \dots \end{aligned} \tag{3.19}$$

Comparing 3.19 with 2.7, we get a system of nonlinear equations as shown in Table 1.

By choosing some parameters as free, we obtain the others which are presented by Table 2.

Table 1: Order condition of (IRK6M)

Order method	Order conditions
First order	$b_1 - b_{-1} = 1$
Second order	$b_{-1} + \sum_{i=1}^{7} b_i = \frac{1}{2}$
Third order	$\sum_{i=2}^{7} b_i c_i = \frac{5}{12}$
Sixth order	$\sum_{i=2}^{7} b_i c_i^4 = \frac{1}{5}$ $\sum_{i=3}^{7} b_i c_i^2 (\sum_{j=2}^{i-1} a_{ij} c_j) = \frac{1}{10}$ $\sum_{i=3}^{7} b_i c_i (\sum_{j=2}^{i-1} a_{ij} c_j^2) = \frac{1}{15}$ $\sum_{i=3}^{7} b_i (\sum_{j=3}^{i-1} c_j (\sum_{k=2}^{j-1} a_{ij} a_{jk}) c_k) = \frac{1}{20}$ $\sum_{i=3}^{7} b_i (\sum_{j=2}^{i-1} a_{ij} c_j^3) = \frac{1}{20}$ $\sum_{i=3}^{7} b_i c_i (\sum_{j=3}^{i-1} (\sum_{k=2}^{j-1} a_{ij} a_{jk}) c_k) = \frac{1}{30}$ $\sum_{i=3}^{7} b_i (\sum_{j=3}^{i-1} (\sum_{k=2}^{j-1} a_{ij} c_j a_{jk}) c_k)) = \frac{1}{40}$ $\sum_{i=3}^{7} b_i (\sum_{j=3}^{i-1} (\sum_{k=2}^{j-1} a_{ij} a_{jk}) c_k^2) = \frac{1}{60}$ $\sum_{i=3}^{7} b_i (\sum_{j=4}^{i-1} (\sum_{k=3}^{j-1} (\sum_{m=2}^{k-1} a_{ij} a_{jk} a_{km}) c_m)) = \frac{1}{120}$

Table 2: Set of coefficients of Butcher's tableau for (IRK6M)

1/3	0.3333						
2/3	0	0.6667					
1/3	0.1101	0.1120	0.1112				
5/6	0.3281	-0.1879	0.5264	0.1667			
1/6	0.5165	0.7894	-0.2493	0.3333	-1.2232		
1	0.2858	-0.1429	0.2755	0.3072	0.1418	0.1326	
$\frac{-187}{400}$	$\frac{213}{400}$	$\frac{-1}{15}$	$\frac{41}{120}$	0	$\frac{83}{300}$	$\frac{-7}{60}$	0

4 Numerical examples

In this section, we test the problem where the IRK6M and RK6M are applied to show the efficiency of the method and the approximate solution compared with the exact solution. The problem is solved for $t \in [0, 1]$.

4.1 Problem 1

$u' = tcosu, u(0) = 1$, (an oscillatory problem), where the exact solution is $u(t) = e^{sint}$.

Table 3: Numerical Solutions Using RK6M, IRK6M of problem 1.

t	Exact	IRK6M	RK6M	$IRKM_{Error}$	$RK6M_{Error}$
0.1	1.1050	1.1048	1.1048	0.0002	0.0002
0.2	1.2198	1.2142	1.2194	0.0056	0.0004
0.3	1.3438	1.3325	1.3432	0.0113	0.0006
0.4	1.4761	1.4589	1.4753	0.0172	0.0008
0.5	1.6151	1.5923	1.6140	0.0228	0.0011
0.6	1.7588	1.7309	1.7574	0.0279	0.0014
0.7	1.9045	1.8726	1.9028	0.0319	0.0017
0.8	2.0490	2.0145	2.0470	0.0345	0.002
0.9	2.1887	2.1536	2.1864	0.0351	0.0023
1.0	2.3198	2.2862	2.3172	0.0336	0.0026

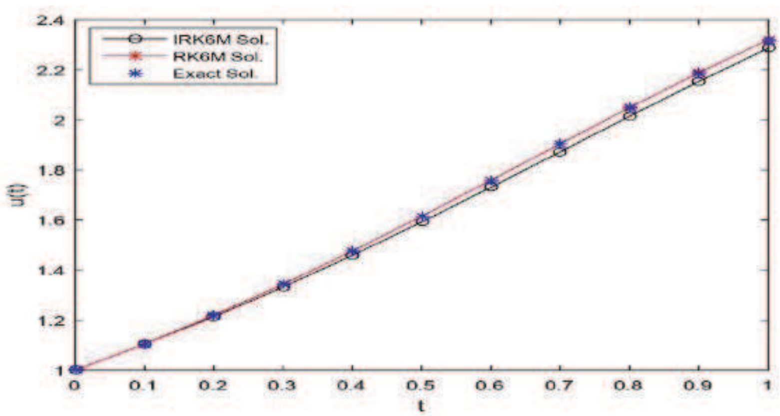

Figure 1: Numerical Solutions of Probem 1

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