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# Generalized-weighted numerical radius inequalities for Schatten $p$-norms

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**Abstract**

In this paper, we present upper and lower bounds for the generalized-weighted numerical radius of $2 \times 2$ matrices for Schatten $p$-norm.

# 1 Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. Let $s_1(A), s_2(A), ..., s_n(A)$ denote the singular values of the matrix $A \in \mathbb{M}_n(\mathbb{C})$; i.e., the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$. For $0 < p < \infty$, define $\|A\|_p$ by $(\sum_{i=1}^n s_i^p(A))^{\frac{1}{p}} = (tr\,|A|^p)^{\frac{1}{p}}$. For $1 \leq p < \infty$, this is the Schatten $p$-norm of $A$.

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ can be presented using the Cartesian Decomposition as $A = Re(A) + iIm(A)$, where $Re(A) = \frac{1}{2}(A + A^*)$ and $Im(A) = \frac{1}{2i}(A - A^*)$ are the real and imaginary parts of $A$, respectively. For $A \in \mathbb{M}_n(\mathbb{C})$ and $0 \leq v \leq 1$, the weighted real and imaginary parts can be defined by $Re_v(A) =$

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$vA+(1-v)A^*$ and $Im_v(A) = v(-iA)+(1-v)(-iA)^*$, respectively. Clearly, $Re_{\frac{1}{2}}(A) = Re(A)$ and $Im_{\frac{1}{2}}(A) = Im(A)$.

Let $N(\cdot)$ be an arbitrary norm on $\mathbb{M}_n(\mathbb{C})$. For every $A \in \mathbb{M}_n(\mathbb{C})$ and unitary $U, V \in \mathbb{M}_n(\mathbb{C})$, the norm $N(\cdot)$ is self-adjoint if $N(A) = N(A^*)$, unitarily invariant if $N(UAV) = N(A)$, and weakly unitarily invariant if $N(U^*AU) = N(A)$. The Schatten $p$-norms $\|\cdot\|_p$ for $p \geq 1$ are typical examples of the self-adjoint and unitarily invariant norms.

If $A, B \in \mathbb{M}_n(\mathbb{C})$, then, for the Schatten $p$-norms, we have the following results from the basic properties of unitarily invariant norms (see [7]) :

$$||A \oplus B||_p = \left\|\left[\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right]\right\|_p = \left(||A||_p^p + ||B||_p^p\right)^{1/p}. \tag{1.1}$$

We refer the reader to [4] for further properties of the the Schatten $p$-norms.

The classical numerical radius of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}. \tag{1.2}$$

It is known that $w(\cdot)$ defines a norm on $\mathbb{M}_n(\mathbb{C})$ and that for every $A \in \mathbb{M}_n(\mathbb{C})$

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.3}$$

The norm $w(\cdot)$ is self-adjoint and weakly unitarily invariant, but it is not unitarily invariant.

A useful identity for the numerical radius was given (see, e.g., [9]) as follows:

$$w(A) = \sup_{\theta\in\mathbb{R}} \left\|Re(e^{i\theta}A)\right\|. \tag{1.4}$$

Several generalizations for the concept of the numerical radius have been discussed in [1], [3], [8], [10], and [11].

In [1], Abu-Omar and Kittaneh defined the generalized numerical radius for every norm $N(\cdot)$ on $\mathbb{M}_n(\mathbb{C})$ and $A \in \mathbb{M}_n(\mathbb{C})$ by

$$w_N(A) = \sup_{\theta\in\mathbb{R}} N\left(Re(e^{i\theta}A)\right). \tag{1.5}$$

In [8], the authors introduced the weighted numerical radius for $0 \leq v \leq 1$ and $A \in \mathbb{M}_n(\mathbb{C})$ as follows:

$$w_v(A) = \sup_{\theta\in\mathbb{R}} \left\|Re_v(e^{i\theta}A)\right\|. \tag{1.6}$$

Very recently, Zamani [11] generalized the weighted numerical radius for $0 \leq v \leq 1$, every norm $N(\cdot)$ on $\mathbb{M}_n(\mathbb{C})$, and $A \in \mathbb{M}_n(\mathbb{C})$, by

$$w_{(N,v)}(A) = \sup_{\theta \in \mathbb{R}} N\left(Re_v(e^{i\theta}A)\right). \tag{1.7}$$

He also proved that if $N(\cdot)$ is weakly unitarily invariant, then so is $w_{(N,v)}(\cdot)$.

In this paper, we let $w_{(p,v)}(\cdot) = \sup_{\theta \in \mathbb{R}} \left\|Re_v(e^{i\theta}A)\right\|_p$ stand for the generalized-weighted numerical radius when $N(\cdot)$ is the Schatten $p$-norm.

In Section 2, we give some inequalities involving the generalized-weighted numerical radius $w_{(p,v)}(\cdot)$ for partitioned general $2 \times 2$ block matrices. In Section 3, our emphasis is on finding such inequalities for the off-diagonal part of block matrices.

# 2 Inequalities involving general $2 \times 2$ block matrices

In this section, we present some inequalities that give upper bounds for $w_{(p,v)}(\cdot)$ for $2 \times 2$ block matrices. Some of these inequalities generalize some inequalities given in [2] and [6].

For the following lemma, see [5].

**Lemma 2.1.** *Let $A \in \mathbb{M}_n(\mathbb{C})$ such that $A = [A_{ij}], 1 \leq i, j \leq n$. Then*

$$n^{2-p}\,||A||_p^p \leq \sum_{i,j=1}^{n} ||A_{i,j}||_p^p \leq ||A||_p^p, \tag{2.1}$$

*for $2 \leq p < \infty$, and*

$$||A||_p^p \leq \sum_{i,j=1}^{n} ||A_{i,j}||_p^p \leq n^{2-p}\,||A||_p^p, \tag{2.2}$$

*for $1 \leq p \leq 2$.*

**Lemma 2.2.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then for $1 \leq p < \infty$ and $0 \leq v \leq 1$, we have*

$$w_{(p,v)}^p\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \sup_{\theta \in \mathbb{R}}\left(\left|\left|ve^{i\theta}A + (1-v)e^{-i\theta}B^*\right|\right|_p^p + \left|\left|(1-v)e^{-i\theta}A^* + ve^{i\theta}B\right|\right|_p^p\right),$$

*for all $\theta \in \mathbb{R}$. In particular,*

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}} w_{(p,v)}(A).$$

*Proof.* We have

$$w^p_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$$

$$\begin{aligned}
&= \sup_{\theta\in\mathbb{R}} \left|\left| Re_v\left(e^{i\theta}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\right|\right|_p^p \\
&= \sup_{\theta\in\mathbb{R}} \left|\left| ve^{i\theta}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} + (1-v)e^{-i\theta}\begin{bmatrix} 0 & B^* \\ A^* & 0 \end{bmatrix}\right|\right|_p^p \\
&= \sup_{\theta\in\mathbb{R}} \left|\left|\begin{bmatrix} 0 & ve^{i\theta}A + (1-v)e^{-i\theta}B^* \\ ve^{i\theta}B + (1-v)e^{-i\theta}A^* & 0 \end{bmatrix}\right|\right|_p^p \\
&= \sup_{\theta\in\mathbb{R}} \left(\left|\left|ve^{i\theta}A + (1-v)e^{-i\theta}B^*\right|\right|_p^p + \left|\left|(1-v)e^{-i\theta}A^* + ve^{i\theta}B\right|\right|_p^p\right).
\end{aligned}$$

□

**Theorem 2.3.** *Let $A = [A_{ij}]$ be a $2 \times 2$ block matrix. Then*

$$w^p_{(p,v)}(A) \le \frac{1}{2^{p-2}}\left(w^p_{(p,v)}\left(\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}\right) + \sum_{i=j} w^p_{(p,v)}(A_{ij})\right), \tag{2.3}$$

*for $p \ge 2$ and $0 \le v \le 1$, and*

$$w^p_{(p,v)}(A) \le w^p_{(p,v)}\left(\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}\right) + \sum_{i=j} w^p_{(p,v)}(A_{ij}), \tag{2.4}$$

*for $1 \le p \le 2$ and $0 \le v \le 1$.*

*Proof.* Using the inequality (2.1) and Lemma 2.2, we get

$$\left|\left|Re_v(e^{i\theta}A)\right|\right|_p^p$$

$$\begin{aligned}
&= \left|\left|\begin{bmatrix} Re_v(e^{i\theta}A_{11}) & ve^{i\theta}A_{12} + (1-v)e^{-i\theta}A_{21}^* \\ ve^{i\theta}A_{21} + (1-v)e^{-i\theta}A_{12}^* & Re_v(e^{i\theta}A_{22}) \end{bmatrix}\right|\right|_p^p \\
&\le \frac{1}{2^{p-2}}\left(\sum_{i=j}\left|\left|Re_v(e^{i\theta}A_{ij})\right|\right|_p^p + \sum_{i\ne j}\left|\left|ve^{i\theta}A_{ij} + (1-v)e^{-i\theta}A_{ji}^*\right|\right|_p^p\right) \quad (2.5)
\end{aligned}$$

Now, the inequality (2.3) follows by taking the supremum over all $\theta \in \mathbb{R}$ on both sides of the inequality (2.5). In a similar manner, we can prove the inequality (2.4) by using the inequality (2.2) and Lemma 2.2. □

Now, we have the following corollary.

**Corollary 2.4.** *Let* $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$ *be a* $2 \times 2$ *block matrix. Then*

$$w^p_{(p,v)}\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) \leq \frac{1}{2^{p-4}} w^p_{(p,v)}(A),$$

*for* $p \geq 2$ *and* $0 \leq v \leq 1$, *and*

$$w^p_{(p,v)}(A) \leq 4 w^p_{(p,v)}(A),$$

*for* $1 \leq p \leq 2$ *and* $0 \leq v \leq 1$.

*Proof.* The proof follows directly from Theorem 2.3 by replacing each $A_{ij}$ by $A$ for all $1 \leq i, j \leq 2$. □

# 3 Inequalities involving off-diagonal $2\times2$ block matrices

In this section, we present some inequalities that give upper and lower bounds for $w_{(p,v)}(\cdot)$ for off-diagonal $2 \times 2$ block matrices. Some of these inequalities generalize some inequalities given in [6].

Our results in this section depend on the following two propositions and cover some basic properties of $w_{(p,v)}(\cdot)$ for the diagonal and off-diagonal $2 \times 2$ block matrices.

**Proposition 3.1.** *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$. *Then for* $1 \leq p < \infty$ *and* $0 \leq v \leq 1$, *we have*

$$w_{(p,v)}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \left(w^p_{(p,v)}(A) + w^p_{(p,v)}(B)\right)^{\frac{1}{p}}.$$

*Proof.* Using equation (1.1), we have

$$\begin{aligned} w_{(p,v)}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left|\left|Re_v\left(e^{i\theta}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right|\right|_p \\ &= \sup_{\theta\in\mathbb{R}} \left|\left|\begin{bmatrix} Re_v\left(e^{i\theta}A\right) & 0 \\ 0 & Re_v\left(e^{i\theta}B\right) \end{bmatrix}\right|\right|_p \\ &= \sup_{\theta\in\mathbb{R}} \left(\left|\left|Re_v\left(e^{i\theta}A\right)\right|\right|^p_p + \left|\left|Re_v\left(e^{i\theta}B\right)\right|\right|^p_p\right) \\ &\leq \left(w^p_{(p,v)}(A) + w^p_{(p,v)}(B)\right)^{\frac{1}{p}}. \end{aligned}$$

□

**Proposition 3.2.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, for $p \geq 1$ and $0 \leq v \leq 1$, we have*

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = w_{(p,v)}\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) \tag{3.1}$$

*and*

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}\right) \tag{3.2}$$

*for all $\theta \in \mathbb{R}$.*

*Proof.* Let $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and $V = \begin{bmatrix} I & 0 \\ 0 & e^{i\theta/2}I \end{bmatrix}$. Then $U$ and $V$ are unitary $2 \times 2$ block matrices. Using the fact that $w_{(p,v)}$ is weakly unitarily invariant, we have

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = w_{(p,v)}\left(U\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U^*\right) = w_{(p,v)}\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right),$$

and

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = w_{(p,v)}\left(V\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}V^*\right) = w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}\right).$$

□

**Theorem 3.3.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, for $p \geq 1$ and $0 \leq v \leq 1$, we have*

$$\begin{aligned} \frac{\max\{w_{(p,v)}(A+B), w_{(p,v)}(A-B)\}}{2^{1-\frac{1}{p}}} &\leq w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \\ &\leq \frac{w_{(p,v)}(A+B) + w_{(p,v)}(A-B)}{2^{1-\frac{1}{p}}}. \end{aligned}$$

*Proof.* Using Lemma 2.2, the triangle inequality, and the inequality (3.1), we have

$$\begin{aligned} 2^{\frac{1}{p}} w_{(p,v)}(A+B) &= w_{(p,v)}\left(\begin{bmatrix} 0 & A+B \\ A+B & 0 \end{bmatrix}\right) \\ &= w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) \\ &\leq w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + w_{(p,v)}\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right) \\ &= 2w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right). \end{aligned}$$

So, we get

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geq \frac{1}{2^{1-\frac{1}{p}}} w_{(p,v)}\left(A+B\right). \tag{3.3}$$

Replacing $B$ by $-B$ in the inequality (3.3), we get

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}\right) \geq \frac{1}{2^{1-\frac{1}{p}}} w_{(p,v)}\left(A-B\right),$$

taking $\theta = \pi$ in the inequality (3.2), we get

$$\begin{aligned} w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}\right) \\ &\geq \frac{1}{2^{1-\frac{1}{p}}} w_{(p,v)}\left(A-B\right), \end{aligned} \tag{3.4}$$

Therefore, by the inequalities (3.3) and (3.4), we get

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \geq \frac{\max\{w_{(p,v)}\left(A+B\right), w_{(p,v)}\left(A-B\right)\}}{2^{1-\frac{1}{p}}},$$

which proves the first inequality.

For the second inequality, consider $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$, where $I$ is the identity matrix. Then $U$ is unitary. Since $w_{(p,v)}$ is weakly unitarily invariant, we have

$$\begin{aligned} 2w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \\ &= 2w_{(p,v)}\left(U\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U^*\right) \\ &= w_{(p,v)}\left(\begin{bmatrix} -(A+B) & A-B \\ -(A-B) & A+B \end{bmatrix}\right) \\ &\leq w_{(p,v)}\left(\begin{bmatrix} -(A+B) & 0 \\ 0 & A+B \end{bmatrix}\right) + w_{(p,v)}\left(\begin{bmatrix} 0 & A-B \\ -(A-B) & 0 \end{bmatrix}\right) \end{aligned} \tag{3.5}$$

By proposition 3.1, we have

$$w_{(p,v)}\left(\begin{bmatrix} -(A+B) & 0 \\ 0 & A+B \end{bmatrix}\right) \leq 2^{\frac{1}{p}} w_{(p,v)}\left(A+B\right). \tag{3.6}$$

By the inequality (3.2) when $\theta = \pi$ and using Lemma 2.2, we have

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A-B \\ -(A-B) & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}} w_{(p,v)}\left(A-B\right). \tag{3.7}$$

Using the inequalities (3.5) and (3.6), and the relation (3.7), we have

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{w_{(p,v)}\left(A+B\right) + w_{(p,v)}\left(A-B\right)}{2^{1-\frac{1}{p}}},$$

which proves the second inequality. □

**Corollary 3.4.** *Let $T \in \mathbb{M}_n(\mathbb{C})$ such that $T = A + iB$, where $A = Re(T)$ and $B = Im(T)$. Then, for $p \geq 1$, we have*

$$\frac{w_{(p,v)}(T)}{2} \leq \frac{1}{2^{\frac{1}{p}}} w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq w_{(p,v)}(T).$$

*Proof.* Replacing $B$ by $iB$ in Theorem 3.3, we get

$$\begin{aligned} \frac{\max\{w_{(p,v)}\left(A+iB\right), w_{(p,v)}\left(A-iB\right)\}}{2^{1-\frac{1}{p}}} &\leq w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix}\right) \\ &\leq \frac{w_{(p,v)}\left(A+iB\right) + w_{(p,v)}\left(A-iB\right)}{2^{1-\frac{1}{p}}} \end{aligned}$$

which is equivalent to

$$\frac{\max\{w_{(p,v)}\left(T\right), w_{(p,v)}\left(T^*\right)\}}{2^{1-\frac{1}{p}}} \leq w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix}\right) \leq \frac{w_{(p,v)}\left(T\right) + w_{(p,v)}\left(T^*\right)}{2^{1-\frac{1}{p}}}.$$

But $w_{(p,v)}\left(T\right) = w_{(p,v)}\left(T^*\right)$. Then

$$\frac{w_{(p,v)}\left(T\right)}{2^{1-\frac{1}{p}}} \leq w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ iB & 0 \end{bmatrix}\right) \leq \frac{w_{(p,v)}\left(T\right)}{2^{-\frac{1}{p}}},$$

The result now follows by taking $\theta = \frac{\pi}{2}$ in the inequality (3.2). □

**Corollary 3.5.** *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, for $2 \leq p < \infty$, we have*

$$w_{(p,v)}\left(\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right) \leq 2w_{(p,v)}(A).$$

*Proof.* Using the inequality (2.1), and the inequality (3.2) when $\theta = \pi$, and applying Lemma 2.2, we have

$$w_{(p,v)}\left(\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right)$$

$$\begin{aligned}
&= \sup_{\theta\in\mathbb{R}} \left|\left|\left| \begin{bmatrix} Re_v(e^{i\theta}A) & ve^{i\theta}A+(1-v)e^{-i\theta}(-A)^* \\ ve^{i\theta}(-A)+(1-v)e^{-i\theta}A^* & -Re_v(e^{i\theta}A) \end{bmatrix} \right|\right|\right|_p \\
&\leq \frac{1}{2^{\frac{2}{p}-1}} \sup_{\theta\in\mathbb{R}} \left(2\left|\left|Re_v(e^{i\theta}A)\right|\right|_p^p + 2\left|\left|ve^{i\theta}A+(1-v)e^{-i\theta}(-A)^*\right|\right|_p^p\right)^{\frac{1}{p}} \\
&\leq \frac{1}{2^{\frac{2}{p}-1}} \left(2w^p_{(p,v)}(A) + w^p_{(p,v)}\left(\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}\right)\right)^{\frac{1}{p}} \\
&= \frac{1}{2^{\frac{2}{p}-1}} \left(2w^p_{(p,v)}(A) + 2w^p_{(p,v)}(A)\right)^{\frac{1}{p}} \\
&= 2w_{(p,v)}(A).
\end{aligned}$$

□

**Theorem 3.6.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, for $2 \leq p < \infty$, we have*

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$$

$$\leq 2^{\frac{1}{p}} \min\{w_{(p,v)}(A), w_{(p,v)}(B)\} + \min\{w_{(p,v)}(A+B), w_{(p,v)}(A-B)\}.$$

*Proof.* Consider $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, where $I$ is the identity matrix, then $U$ is unitary.

Using the triangle inequality, Corollary 3.5, the inequality (3.2) when $\theta = \pi$, and Lemma 2.2, we have

$$w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$$

$$
\begin{aligned}
&= w_{(p,v)}\left(U\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U^*\right)\\
&= \frac{1}{2}w_{(p,v)}\left(\begin{bmatrix} A+B & A-B \\ -(A-B) & -(A+B) \end{bmatrix}\right)\\
&= \frac{1}{2}w_{(p,v)}\left(\begin{bmatrix} A+B & A+B \\ -(A+B) & -(A+B) \end{bmatrix}+\begin{bmatrix} 0 & -2B \\ 2B & 0 \end{bmatrix}\right)\\
&\leq \frac{1}{2}\left(w_{(p,v)}\left(\begin{bmatrix} A+B & A+B \\ -(A+B) & -(A+B) \end{bmatrix}\right)+w_{(p,v)}\left(\begin{bmatrix} 0 & -2B \\ 2B & 0 \end{bmatrix}\right)\right)\\
&\leq \frac{1}{2}\left(2w_{(p,v)}\left(A+B\right)+2^{\frac{1}{p}}w_{(p,v)}\left(2B\right)\right)\\
&= w_{(p,v)}\left(A+B\right)+2^{\frac{1}{p}}w_{(p,v)}\left(B\right). \qquad (3.8)
\end{aligned}
$$

Replacing $B$ by $-B$ in (3.8) and using the inequality (3.2) when $\theta=\pi$, we get

$$
w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq w_{(p,v)}\left(A-B\right)+2^{\frac{1}{p}}w_{(p,v)}\left(B\right). \qquad (3.9)
$$

Then, by the inequality (3.8) and the inequality (3.9), we get

$$
w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}}w_{(p,v)}\left(B\right)+\min\{w_{(p,v)}\left(A+B\right), w_{(p,v)}\left(A-B\right)\}. \qquad (3.10)
$$

Using the inequality (3.1) and the inequality (3.10), we have

$$
w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq 2^{\frac{1}{p}}w_{(p,v)}\left(A\right)+\min\{w_{(p,v)}\left(A+B\right), w_{(p,v)}\left(A-B\right)\}. \qquad (3.11)
$$

Then by the inequality (3.10) and the inequality (3.11), we get

$$
\begin{aligned}
&w_{(p,v)}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)\\
&\leq 2^{\frac{1}{p}}\min\{w_{(p,v)}\left(A\right), w_{(p,v)}\left(B\right)\}+\min\{w_{(p,v)}\left(A+B\right), w_{(p,v)}\left(A-B\right)\},
\end{aligned}
$$

as required. □

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