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Analytical Solution to a Coupled Pendula

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Abstract

In this paper, we give an analytical solution to a system of two coupled pendula. The obtained solutions are valid for large deviation angles. We illustrate the results in a concrete example.

1 Introduction.

In nature there are many examples of the coupled pendulum. The nature of the physics is governed mainly by the coupling constant between two pendula, leading to the complicated physical behaviors. Two pendula, connected by a mass-less spring, are allowed to vibrate in the same vertical plane. Two identical pendula of length L and mass m with inetrtial moment J are connected through a weak spring of spring constant k . Figure 1 shows the schematic diagram of the system.

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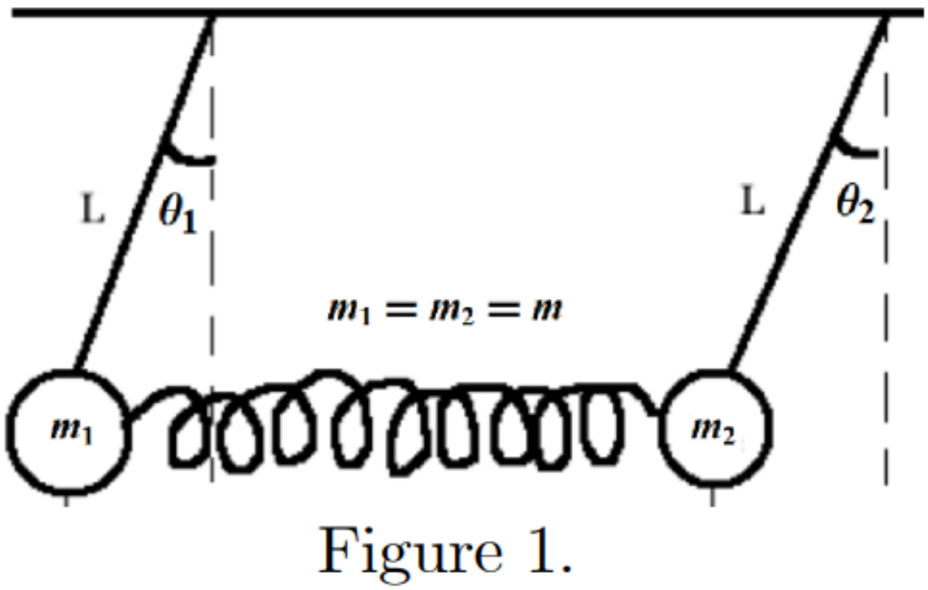

Figure 1.

The equation of motion of the combined system is then given by:

$$\begin{aligned} J\,\ddot{\theta}_1 &= -mLg\sin\theta_1 - k(\sin\theta_1 - \sin\theta_2). \\ J\,\ddot{\theta}_2 &= -mLg\sin\theta_2 + k(\sin\theta_1 - \sin\theta_2). \end{aligned} \tag{1.1}$$

Our aim is to solve this system analytically given the initial conditions

$$\theta_1(0) = \alpha,\ \theta_1'(0) = \dot{\alpha},\ \theta_2(0) = \beta,\ \theta_2'(0) = \dot{\beta}. \tag{1.2}$$

2 Main results

We will solve the problem described by (1.1)-(1.2). To this end, we first use the approximation $\sin\theta \approx \theta - \rho\theta^3/6$, where ρ is a free parameter that we choose in order to get as less residual error as possible. The default ρ value is $\rho = 1$. Then the system (1.1) takes the form

$$\begin{aligned} J\ddot{\theta}_1 + mgL\left(\theta_1 - \frac{\theta_1^3}{6}\right) + k\left(\theta_1 - \theta_2 - \rho\frac{\theta_1^3}{6} + \rho\frac{\theta_2^3}{6}\right) &= 0. \\ J\ddot{\theta}_2 + mgL\left(\theta_2 - \frac{\theta_2^3}{6}\right) - k\left(\theta_1 - \theta_2 - \rho\frac{\theta_1^3}{6} + \rho\frac{\theta_2^3}{6}\right) &= 0. \end{aligned} \tag{2.3}$$

Next, we make the substitution $\theta_1 = u + v$ and $\theta_2 = u - v$, $u = u(t)$ and $v = v(t)$, where the functions u and v are the solutions to some Duffing equations

$$\ddot{u}+pu+qu^3=0 \text{ and } \ddot{v}+rv+sv^3=0. \tag{2.4}$$

Then

$$\begin{aligned}&\left(\tfrac{mgL}{J}+\tfrac{2k}{J}-r\right)v+\left(-\tfrac{mgL\rho}{6J}-\tfrac{k\rho}{3J}-s\right)v^3+\\ &\left(-\tfrac{mgL\rho}{2J}-\tfrac{k\rho}{J}\right)u^2v+\left(-\tfrac{mgL\rho v^2}{2J}+\tfrac{mgL}{J}-p\right)u+\\ &u^3\left(-\tfrac{mgL\rho}{6J}-q\right)=0.\\ &\left(-\tfrac{mgL}{J}-\tfrac{2k}{J}+r\right)v+\left(\tfrac{mgL\rho}{6J}+\tfrac{k\rho}{3J}+s\right)v^3+\\ &\left(\tfrac{mgL\rho}{2J}+\tfrac{k\rho}{J}\right)u^2v+\left(-\tfrac{mgL\rho v^2}{2J}+\tfrac{mgL}{J}-p\right)u+\\ &\left(-\tfrac{mgL\rho}{6J}-q\right)u^3=0.\end{aligned} \tag{2.5}$$

The system (2.5) suggests the following choices: Solving the above system gives

$$p=\frac{mgL}{J},\ q=-\frac{\rho mgL}{6J},\ r=\frac{mgL+2k}{J},\ s=-\frac{\rho(mgL+2k)}{6J}. \tag{2.6}$$

Then

$$\begin{aligned}&J\,\ddot{\theta}_1+mLg\sin\theta_1+k(\sin\theta_1-\sin\theta_2)\approx-\tfrac{\rho}{2J}uv(mgL(u+v)+2ku).\\ &J\,\ddot{\theta}_2+mLg\sin\theta_2-k(\sin\theta_1-\sin\theta_2)\approx\tfrac{\rho}{2J}uv(mgL(u-v)+2ku).\end{aligned}$$

We expect good approximate solutions for small m and k. The functions u and v must satisfy the following initial conditions:

$$u(0)=\frac{\alpha+\beta}{2},u'(0)=\frac{1}{2}\left(\dot{\alpha}+\dot{\beta}\right),v(0)=\frac{\alpha-\beta}{2},v'(0)=\frac{1}{2}\left(\dot{\alpha}-\dot{\beta}\right). \tag{2.7}$$

Thus,

$$\ddot{u}+\frac{mgL}{J}u-\frac{\rho mgL}{6J}u^3=0,\ u(0)=\frac{\alpha+\beta}{2},u'(0)=\frac{1}{2}\left(\dot{\alpha}+\dot{\beta}\right). \tag{2.8}$$

$$\ddot{v}+\frac{mgL+2k}{J}v-\frac{\rho(mgL+2k)}{6J}v^3=0,\ v(0)=\frac{\alpha-\beta}{2},v'(0)=\frac{1}{2}\left(\dot{\alpha}-\dot{\beta}\right). \tag{2.9}$$

These last two odes admit an exact solution. More precisely, the solution to the i.v.p.

$$\ddot{w}+pw+qw^3=0,\ w(0)=w_0 \text{ and } w'(0)=\dot{w}_0 \tag{2.10}$$

is written as

$$\begin{aligned}&w=w(t)=\frac{w_0\operatorname{cn}\left(\sqrt{\omega}t\middle|m\right)+\frac{\dot{w}_0}{\sqrt{\omega}}\operatorname{dn}\left(\sqrt{\omega}t\middle|m\right)\operatorname{sn}\left(\sqrt{\omega}t\middle|m\right)}{1+\frac{p+qw_0^2-\omega}{2\omega}\operatorname{sn}\left(\sqrt{\omega}t\middle|m\right)^2}.\\ &\omega=\sqrt{(p+qw_0^2)+2q\dot{w}_0^2},\ m=\tfrac{1}{2}-\tfrac{p}{2((p+qw_0^2)+2q\dot{w}_0^2)}\end{aligned} \tag{2.11}$$

In particular, an approximate analytical solution to the i.v.p

$$\begin{aligned} J\,\ddot{\theta}_1 &= -mLg\sin\theta_1 - k(\sin\theta_1 - \sin\theta_2),\ \theta_1(0)=\alpha \ \text{ and } \theta_1'(0)=0. \\ J\,\ddot{\theta}_2 &= -mLg\sin\theta_2 + k(\sin\theta_1 - \sin\theta_2),\ \theta_2(0)=\beta \text{ and } \theta_1'(0)=0. \end{aligned} \tag{2.12}$$

will be

$$\theta_1(t) = u(t) + v(t) \text{ and } \theta_2(t) = u(t) - v(t), \tag{2.13}$$

where

$$\begin{aligned} u(t) &= \tfrac{1}{2}(\alpha+\beta)\mathrm{cn}\left(\sqrt{\omega}t \mid \mu\right). \\ v(t) &= \tfrac{1}{2}(\alpha-\beta)\mathrm{cn}\left(\sqrt{w}t \mid \kappa\right). \\ \omega &= \tfrac{mgL\left(24-\rho(\alpha+\beta)^2\right)}{24J},\ \ \mu = \tfrac{\rho(\alpha+\beta)^2}{\rho(\alpha+\beta)^2-24}. \\ w &= \tfrac{\left(24-\rho(\alpha-\beta)^2\right)(mgL+2k)}{24J},\ \ \kappa = \tfrac{\rho(\alpha-\beta)^2}{\rho(\alpha-\beta)^2-24}. \end{aligned} \tag{2.14}$$

The functions u and v are periodic and their periods read

$$\begin{aligned} &T_u = 4/\sqrt{\omega}K\left(\mu\right) \text{ and } T_v = 4/\sqrt{w}K\left(\kappa\right). \\ &K(m) \approx \tfrac{\pi(5m-16)}{18m-32} \text{ for small } m. \end{aligned} \tag{2.15}$$

The Jacobian Function $\mathrm{cn}(t,m)$ with modulus m and parameter $k=\sqrt{m}$ may be approximated for $-1 \leq m \leq 0.5$ as follows :

$$\mathrm{cn}(t,m) \approx \frac{\sqrt{1+\lambda}\cos\left(\sqrt{1+\lambda}t\right)}{\sqrt{1+\lambda\cos^2\left(\sqrt{1+\lambda}t\right)}},\ \lambda = \frac{1}{14}\left(\sqrt{m^2-144m+144}-m-12\right). \tag{2.16}$$

Yet another approximation is:

$$\begin{aligned} &\mathrm{cn}(t,m) \approx (1-\upsilon)\cos\left(\sqrt{\tfrac{1}{8\upsilon+1}}t\right) + \upsilon\cos\left(3\sqrt{\tfrac{1}{8\upsilon+1}}t\right), \\ &\text{where } \ \upsilon = -\tfrac{24m(2m-3)}{413m^2-1512m+1152}. \end{aligned} \tag{2.17}$$

The period of the oscillations T is obtained from the equation $T = pT_u \approx qT_v$, for some positive integers that we will choose the least in magnitude as possible.

3 Analysis and Discussion

Let us analyze the accuracy of the obtained solutions. In the case when

$$\theta_1(0) = \theta_2(0) = \alpha,\ \theta_1'(0) = \ \theta_2'(0) = \dot{\alpha}\ , \tag{3.18}$$

the system behaves like a single pendulum and then $\theta_1 = \theta_2 = \theta$. The problem reduces to

$$J\,\ddot{\theta} = -mgL\sin\theta, \theta(0) = \alpha \text{ and } \theta'(0) = \dot{\alpha}. \tag{3.19}$$

Now, assuming that

$$\theta_1(0) = -\theta_2(0) = \alpha,\ \theta_1'(0) = \ -\theta_2'(0) = \dot{\alpha}\ , \tag{3.20}$$

we may set $\theta_2 = -\theta_1$ and then we must solve the i.v.p.

$$J\,\ddot{\theta} = -mgL\sin\theta - 2k\sin\theta,\ \theta(0) = \alpha \text{ and } \theta'(0) = \dot{\alpha} \tag{3.21}$$

The two problems (3.20) and (3.21) have the form

$$\ddot{\theta} + \omega_0^2\sin\theta = 0,\ \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0. \tag{3.22}$$

The exact solution to the i.v.p. (3.22) may be found in [1].

Example. Let $\alpha = 30°$, $\beta = 0$, $J = L = m = k = 1$ and $g = 9.8$ (See Figure 2).

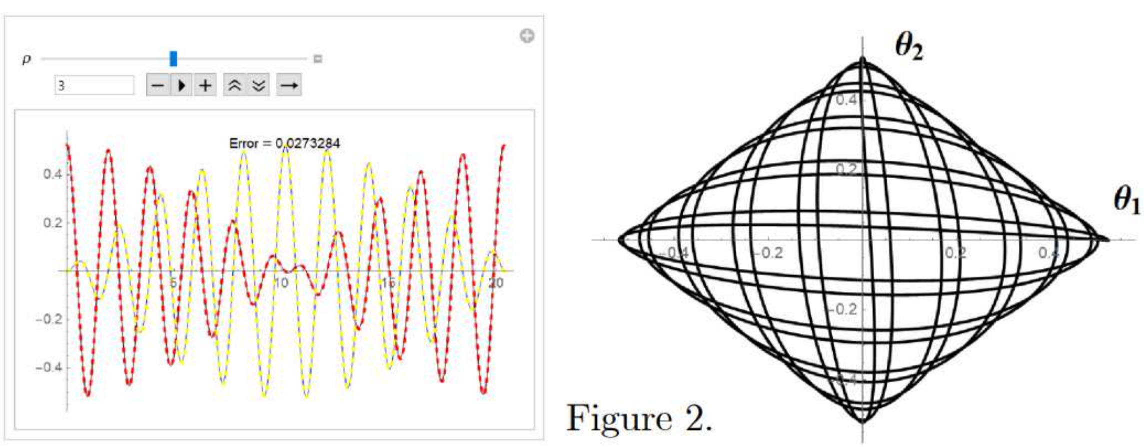

Figure 2.

The approximate solutions are:

$$\theta_1(t) = 15°\text{cn}(3.07639t| - 0.0177428) + 15°\text{cn}(3.37574t| - 0.0177428).$$
$$\theta_2(t) = 15°\text{cn}(3.07639t| - 0.0177428) - 15°\text{cn}(3.37574t| - 0.0177428)$$

$$\theta_1(t) = \frac{0.983883\cos(3.08995t)}{\sqrt{14+0.123744\cos^2(3.08995t)}} + \frac{0.983883\cos(3.39063t)}{\sqrt{14+0.123744\cos^2(3.39063t)}}.$$
$$\theta_2(t) = \frac{0.983883\cos(3.08995t)}{\sqrt{14+0.123744\cos^2(3.08995t)}} - \frac{0.983883\cos(3.39063t)}{\sqrt{14+0.123744\cos^2(3.39063t)}}.$$

The periods of the functions u and v are $T_u = 2.03342$ and $T_v = 1.8531$ and we have $10T_1 \approx 11T_2$ so that we get the following estimation for the period of the oscillations: $T = 1/2(10T_1+11T_2) = 20.35916$. The exact period obtained using the Runge-Kutta numerical solution reads $T_{\text{exact}} = 20.35593036$ and so our estimation is very good.

References

[1] Alvaro H. Salas, Analytic solution to the pendulum equation for a given initial conditions. Journal of King Saud University-Science, **32,** no. 1, (2020), 974–978.

[2] A. Nayfeh, D. Mook, Nonlinear Oscillations, Wiley Classics Library, 1995.