International Journal of Mathematics and
Computer Science, **17**(2022), no. 3, 1475–1481

On tori D_0 in $E_6(K)$ for fields K of characteristic two

AbdulKarem Alhuraiji, Mashhour Al-Ali Bani-Ata

Department of Mathematics
The Public Authority for Applied Education and Training
Adailiyah, Kuwait

email: AbdulKaremalhuraiji@gmail.com, mashhour_ibrahim@yahoo.com

(Received April 7, 2022, Accepted May 14, 2022)

Abstract

The purpose of this paper is to give an explicit and elementary description for a torus D_0 in the Chevalley group $E_6(K)$ for fields K of characteristic two, using the notion of root bases introduced in [2] and the generalized quadrangle $(\mathbb{P}, Ł)$. We use this construction to show that $M = \langle W, D_0\rangle$ is Fischer embedded, or a 3-transposition group. Among other results we investigate the action of the Weyl group of type E_6 on the torus D_0. For more information about tori in $E_6(q^*)$, see [4].

1 General setup and notations

We start with the generalized quadrangle $(\mathbb{P}, \mathcal{L})$ of type $O_6^-(2)$, with points set $\mathbb{P}$ and lines set $\mathcal{L}$; i.e., we consider (V, Q), where V is a 6-dimensional vector space over $\mathbb{F}_2$ and Q a non-degenerate quadratic form on V of minimal Witt-index, $\mathbb{P} = \{0 \neq x \in V \mid Q(x) = 0\}$, $\mathcal{L} = \{L < V \mid Q(L) = 0,\ \dim L = 2\}$, $W = \bar{O}_6(V, Q) = \{g \in GL(V) \mid Q(x^g) = Q(x),\ \forall x \in V\}$. W is of the type $\Omega_6^-(2) \rtimes 2$ or $U_4(2) \rtimes 2$ and it is a 3-transposition group generated by the 36-reflections $\sigma_s : x \to x + (x|s)s$, where $(\cdot\,|\,\cdot)$ is the bilinear form corresponding to Q defined by $(x|y) = Q(x + y) + Q(x) + Q(y)$.

Key words and phrases: Generalized quadrangle, E-set, M-set, root involution, Fischer embedded, torus.
AMS (MOS) Subject Classifications: 17A75.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

Definition 1.1. An M-set Δ is a subset of $\mathbb{P}$ of size 6 such that $(x|y) = 1$ for all $x, y \in \Delta$ and $x \neq y$.

Proposition 1.1. *[2] Let Δ be an M-set. Set $s = s_\Delta = \sum_{x \in \Delta} x$. Then*

1. *$Q(s_\Delta) = 1$ and $\Delta^* = \Delta^{\sigma_s} = \Delta + s_\Delta$ is also an M-set.*
2. *$\mathbb{P} = \Delta \cup \Delta^* \cup \Delta_0$ where $\Delta_0 = \{x \in \mathbb{P} \mid (x|s_\Delta) = 0\}$.*

Definition 1.2. Let K be a field of characteristic two and let A be a 27-dimensional vector space over K with basis $e_x, x \in \mathbb{P}$. On A we define a trilinear form T and a distributive multiplication by

$$T(e_x, e_y, e_z) = \begin{cases} 1 & , \quad (x,y,z) \in \mathcal{L} \\ 0 & , \quad otherwise \end{cases} \qquad \text{and } e_x e_y = \begin{cases} e_z & , \quad (x,y,z) \in \mathcal{L} \\ 0 & , \quad otherwise \end{cases}$$

Definition 1.3. For an M-set Δ and $k \in K$, define root-elements $r_\Delta(k) \in GL(A)$ by:

$$r_\Delta(k) = \begin{cases} e_x + k e_{x^{\sigma_\Delta}} & , \quad x \in \Delta \\ e_x & , \quad otherwise \end{cases} , \text{ where } \sigma_\Delta = \sigma_{s_\Delta} = \sigma_s.$$

The group $E(K) = \langle r_\Delta(k) \mid \Delta \text{ is } M-set, k \in K \rangle$ modulo its center is the Chevalley group of type E_6.

Definition 1.4. We define a quadratic map $\hat{Q} : A \to A$ by $\hat{Q}(a) = \sum Q_p(a) e_p$ where $Q_p(a) = \sum_{(p,x,y) \in \mathcal{L}} a_x a_y$ and $a = \sum_{x \in \mathbb{P}} a_z e_x$.

Definition 1.5. Define $G_0 = \{g \in GL(A) | \ \hat{Q}(a^g) = \hat{Q}(a)^g \ , \ \forall a \in A\}$, where $g^* = (g^t)^{-1}$ with respect to the basis $e_x, x \in A$.

Proposition 1.2. *[3] The group G_0 is an intermediate group between $E(K)$ and the group of isometries of T.*

Definition 1.6. Let $D_0 = \{d \in G_0 | \ e_x^d = d_x e_x \ , \ \forall x \in \mathbb{P}\}$. D_0 is called a torus. Define $\hat{D} = \{d \in GL(A) \mid e_x^d = d_x e_x \ \forall x \in \mathbb{P} \ , \ d_x \neq 0$ and there exists $k = k_d$ such that $\prod_{x \in L} d_x = k$ for all $L \in \mathcal{L}\}$.

Definition 1.7. Let $\mathsf{a} = \{a \in A \mid a \neq 0 \ , \ (a|a) = 0 \ , \ \hat{Q}(a) = 0\}$. Define $\mathsf{a}_2 = \{e_p + k e_Q \mid (p|Q) = 1 \ , \ k\bar{k} = 1\}$.

Definition 1.8. Let $a \in A$ such that $a = \sum a_x e_x$. Then *support*$(a) = \{x \in \mathbb{P} \mid a_x \neq 0\}$.

Definition 1.9. Let K admits an automorphism α of order 2, written as $k^\alpha = \bar{k}$ for $kinK$ and let $(\ |\ )$ be a unitary form corresponding to α, defined by $(e_x|e_y) = \begin{cases} 1 & , \quad x = y \\ 0 & , \quad \text{otherwise} \end{cases}$, or $(a|b) = \sum_x a_x \bar{b}_x$ for $a = \sum_{x\in\mathbb{P}} a_x e_x$, $b = \sum_{x\in\mathbb{P}} b_x e_x$.

Definition 1.10. Define $U(K) = \{g \in G_0 \mid g \text{ preserves } (\cdot|\cdot)\}$.

2 Results

Proposition 2.1. *The Weyl group W normalizes D_0.*

Proof. As W acts on A by $e_x^g = e_{x^g}$ for $g \in W$, then if $d \in D_0$, we have:

$$e_x^{d^g} = e_x^{g^{-1}dg} = (e_{xg^{-1}})^{dg} = (d_{xg^{-1}} e_{xg^{-1}})^g = d_{xg^{-1}} e_x.$$

Let $d^g = d'$ with $d'_x = d_{xg^{-1}}$ and let $L \in \mathcal{L}$ be a line. Then $\prod_{x\in L} d'_x = \prod_{x\in L} d_{xg^{-1}} = \prod_{y\in Lg^{-1}} d_y = 1$, Hence, $d' \in D_0$. □

Theorem 2.1. *Let $d \in \hat{D}$ with $k = k_d$ as above, and let Δ be a M-set with $x, y \in \Delta$, $x \neq y$, corresponding to the reflection $\sigma = \sigma_s = \sigma_{s_\Delta}$. Then*

1. *If $\{x, y^\sigma, x+y^\sigma\}$ is a transversal line, then $d_{y^\sigma} = m d_y$ for some $m \in K^*$.*
2. *If $\{x_1 + x_2^\sigma, x_3 + x_4^\sigma, x_5 + x_6^\sigma\}$ is a line corresponding to a partition $\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}$ of Δ, then $\prod_{x\in\Delta} d_x = \dfrac{k^2}{m^3}$.*

Proof. As the point $x + y^\sigma = x^\sigma + y$ is on two transversal lines namely $\{x, y^\sigma, x + y^\sigma\}, \{y, x^\sigma, y + x^\sigma\}$. Hence, $k = d_x d_{y^\sigma} d_{x+y^\sigma} = d_y d_{x^\sigma} d_{x+y^\sigma}$. This implies $d_{y^\sigma} = d_y d_x^{-1} d_{x^\sigma}$, $\forall x \neq y$ or there exists $m \in K^*$ with $d_{y^\sigma} = m \cdot d_y$ and $d_{x+y^\sigma} = \dfrac{k}{m d_x d_y}$, and as $\{x_1 + x_2^\sigma, x_3 + x_4^\sigma, x_5 + x_6^\sigma\}$ is a line, then $k = d_{x_1+x_2^\sigma} d_{x_3+x_4^\sigma} d_{x_5+x_6^\sigma} = \dfrac{k^3}{m^3 \prod_{x\in\Delta} d_x}$. This implies $\prod_{x\in\Delta} d_x = \dfrac{k^2}{m^3}$ and the claim. □

Remark 2.1. *In particular, if $K = \mathbb{F}_4$, then $\hat{D} = D_0 \cup D_1 \cup D_2$ where $D_i = \{d \in \hat{D} \mid \prod_{x\in L} d_x = w^i\ ,\ \forall L \in \mathcal{L}\ ,\ w^3 = 1\}$, for a M-set Δ and $d \in D_i$, it holds $\prod_{x\in\Delta} d_x = w^{-i}$.*

Theorem 2.2. $|D_0| = |D_1| = |D_2| = 3^6$ *and* $|\hat{D}| = 3^7$.

Proof. Let $H = \{diag(d_x \mid x \in \Delta | d_x^3 = 1)\}$ and $\rho : D_0 \longrightarrow H$ be the restriction of D_0 to $\langle e_x | x \in \Delta \rangle$ i.e. $\rho(d) = (d_x)_{x \in \Delta}$ with $\ker \rho = \hat{N} = \{d \in D_0 \mid d_x = 1 \ \forall x \in \Delta\}$. So let $x, y \in \Delta$. Then $\{x, y + s_\Delta, x + y + s_\Delta\}$ and $\{y, x + s_\Delta, y + x + s_\Delta\}$ are two transversal lines intersect at $p = x + y + s_\Delta$. If $\rho \in \hat{N} \leqslant D_0$, it follows that $1 = d_x d_{y+s_\Delta} d_p = d_{y+s_\Delta} d_p$ and $1 = d_y d_{x+s_\Delta} d_p = d_y d_{x+s_\Delta}$ as $d_y = d_x = 1$ for $x, y \in \Delta$. Hence, it follows that $d_{x+s_\Delta} = d_{y+s_\Delta}$ which means $d_x = d_y$, $\forall x, y \in \Delta^*$. Hence, there exists $k \in K^*$ such that $d_x = \begin{cases} 1 & , \ x \in \Delta \\ k & , \ x \in \Delta^* \\ k^{-1} & , \ x \in \Delta_0 \end{cases}$. This implies that $|\ker \rho| = 3$ and hence $|D_0| = |\text{image}\,\rho||\ker \rho|$ where $\text{image}\,\rho$ consists of all 6×6 diagonal matrices of determinant 1, by Theorem 2.1. Hence, $|D_0/\hat{N}| = |\mathbb{F}_4^*|^5$ and $|D_0| = 3^6$. □

Theorem 2.3. *Let* Δ *be a* M*-set corresponding to* $s = s_\Delta$ *and the reflection* $\sigma = \sigma_s = \sigma_{\Delta_s}$. *Then* $\sigma_s^{D_0} = \{\sigma_s(k) \ |0 \neq k \in \mathbb{F}_4\}$ *and* $|\bigcup_{\substack{s \in V \\ Q(s)=1}} \sigma_s^{D_0}| = 108$.

Proof. The map $\rho : D_0 \to D_0$ defined by $\rho(d) = d^{-1} d^\sigma$ is a homomorphism and $e_x^{\rho(d)} = (d^{-1} d^\sigma)_x e_x = d_x^{-1} d_{x^\sigma} e_x$ for all $x \in \mathbb{P}$. As $x, y \in \Delta$, $x \neq y$, then $x + y^\sigma$ is on the lines $\{x + y^\sigma, x, y^\sigma\}$ and $\{x + y^\sigma, y, x^\sigma\}$. Hence $1 = d_{x+y^\sigma} d_x d_{y^\sigma} = d_{x+y^\sigma} d_y d_{x^\sigma}$ and this implies $d_x d_{y^\sigma} = d_y d_{x^\sigma}$ or $d_x^{-1} d_{x^\sigma} = d_y^{-1} d_{y^\sigma}$. Hence, there exists $k \in \mathbb{F}_4^*$, such that $(d^{-1} d^\sigma)_x = d_x^{-1} d_{x^\sigma} = \begin{cases} k & , \ x \in \Delta \\ k^{-1} & , \ x \in \Delta^* \\ 1 & , \ x \in \Delta_0 \end{cases}$

This implies $|\text{image}\,\rho| = 3$ .....(∗).
As

$$(e_x)\sigma^d = e_x^{d^{-1}\sigma d} = (d_x^{-1} e_x)^{\sigma d} = (d_x^{-1} e_{x^\sigma})^d = d_x^{-1} d_{x^\sigma} e_{x^\sigma} = k e_{x^\sigma} = \begin{cases} k e_{x^\sigma} & , \ x \in \Delta \\ k^{-1} e_{x^\sigma} & , \ x \in \Delta^* \\ e_{x^\sigma} & , \ x \in \Delta_0 \end{cases}$$

.....(∗∗)
From (∗) and (∗∗) as there are 36 reflections and the claim follows. □

Proposition 2.2. *[2] If* $h \in End_K(A)$, *such that* $e_x^h = e_x + \sum_{y \in \mathbb{P}} (x|y) e_y$ *for all* $x \in \mathbb{P}$, *then*

(a) *h is an involution and* $\dim [A, h] = 6$, *where* $[A, h] =< a^h - a \mid a \in A >$.

(b) $h \in U(K)$.

Theorem 2.4. *D_0 is transitive on h^{D_1} and on h^{D_2}.*

Proof. First we show that $|C_{D_0}(h)| = |wI|$ where $w^3 = 1$. Let $d \in C_{D_0}(h)$, it follows that $e_x^h = e_x^{d^{-1}hd}$, $\forall x \in \mathbb{P}$, $e_x^{d^{-1}hd} = d_x^{-1} e_x^{hd} = d_x^{-1}(e_x + \sum\limits_{(y|x)=1} e_y)^d = d_x^{-1}(d_x e_x + \sum\limits_{(x|y)=1} d_y e_y) = e_x + \sum\limits_{(y|x)=1} d_x^{-1} d_y e_y = e_x + \sum\limits_{(y|x)=1} e_y$ if and only if $d_x = d_y$ for all x and y in $\mathbb{P}$ with $(x|y) = 1$. Let Δ be a M-set, then $d_x = d_y$ for all $x, y \in \Delta$ and $d_x = d_{x^\sigma}$ for all $x \in \Delta$. Hence there exists $k \in \mathbb{F}_4^*$ such that $d_x = k$, $\forall x \in \Delta \cup \Delta^\sigma$. If $y \in \Delta_0$, then $y = x_1 + x_2$ for $x_1 \in \Delta$ and $x_2 \in \Delta^*$. As $\prod\limits_{x \in L} d_x = 1$, then it follows that $d_y = d_{x_1}^{-1} d_{x_2}^{-2} = k^{-2} = k$ as $\{y, x_1, x_2\} = L \in \mathcal{L}$. This implies $|C_{D_0}(h)| = |wI| = 3$ and $|h^{D_0}| = \dfrac{|D_0|}{|C_{D_0}(h)|} = 3^5$, and as $|C_{D_0}(h^{d_i})| = |C_{D_0}(h)|$ for $i = 1, 2$, the claim follows. □

Definition 2.1. For $s \in V$ with $Q(s) = 1$, let Δ and Δ^* be the corresponding M-sets to S and Δ_0 be the set of points orthogonal to s. Define τ_s by

$$e_x^{\tau_s} = \begin{cases} e_x + \sum\limits_{y \in \Delta} e_y & , \quad x \in \Delta \\ e_x + \sum\limits_{y \in \Delta^*} e_y & , \quad x \in \Delta^* \\ e_x + \sum\limits_{y \in \Delta_0} (x|y) e_y & , \quad x \in \Delta_0 \end{cases}$$

Remark 2.2. *[2] Let Δ be a M-set corresponding to s with $Q(s) = 1$, and let $e_\Delta = \sum\limits_{x \in \Delta} e_x$, then $\tau_s = t_{e_\Delta} = t_{e_{\Delta^*}}$, where $e_x^{t_{e_\Delta}} = e_x + (e_x e_\Delta) e_\Delta + (e_x | e_\Delta) e_\Delta$. If $x \in \Delta$, then $e_x^{t_{e_\Delta}} = e_x + e_y$.*

Proposition 2.3. *Let Δ be a M-set corresponding to $s = \sum\limits_{x \in \Delta} x$. Then $\tau_s^{D_0} = \{t_a \mid a \in \boldsymbol{a}_\Delta\}$.*

Proof. As $\tau_s = t_{e_\Delta}$, where $e_\Delta = \sum\limits_{x \in \Delta} e_x$, then by Lemma 3.1[2] $\tau_s^d = t_{e_\Delta^d}$ for $d \in D_0$ and $e_\Delta^d = \sum\limits_{x \in \Delta} e_x^d = \sum\limits_{x \in \Delta} d_x e_x = a = \sum\limits_{x \in \Delta} a_x e_x$, where $support(a) = \Delta$, $\prod\limits_{x \in support(a)} d_x = \prod\limits_{x \in \Delta} d_x = 1$. Hence $\tau_s^{D_0} = \{t_a \mid support(a) = \Delta \text{ , } \prod\limits_{x \in \Delta} a_x = 1\}$ and $\tau_s^{D_i} = \{t_b \mid support(b) = \Delta \text{ , } \prod\limits_{x \in \Delta} b_x = w^i \text{ , } i = 1, 2\}$. Hence the claim. □

Theorem 2.5. *The set of long roots $\{t_a \mid a \in \boldsymbol{a}_2\}$ generates the monomial group $M = D_0 \rtimes \hat{W}$, where $\hat{W}$ is the Weyl group of type $O_6^-(V, Q)$.*

Proof. Let $a \in \mathsf{a}_2$, $a = e_P + ke_Q$ for $P, Q \in \mathbb{P}$, $(P, Q) = 1$ and $k\bar{k} = 1$. Then
$e_P^{t_a} = e_P + (e_P a)\bar{a} + \langle e_P | a \rangle a = e_P + a = ke_Q$
$e_Q^{t_a} = e_Q + (e_Q a)\bar{a} + \langle e_Q | a \rangle a = e_Q + \bar{k}a = \bar{k}e_P$
If $x \neq P, Q$, then $e_x^{t_a} = e_x + (e_x a)\bar{a}$ and consider the following cases

1. If $(P|x) = (Q|x) = 1$, then $e_x a = 0$.
2. If $(P|x) = 0$ and $(Q|x) = 1$, then $e_x a = (e_x e_P)(e_P + \bar{k}e_Q) = e_x + \bar{k}e_{x+P+Q}$.
3. If $(P|x) = 1$ and $(Q|x) = 0$, then $(e_x a)\bar{a} = k(e_x e_Q)(e_P + \bar{k}e_Q) = ke_{x+P+Q} + k\bar{k}e_x = e_x + ke_{x+P+Q}$.
4. If $(P|x) = (Q|x) = 0$, then $e_x a = e_x(e_P + ke_Q) = e_{x+P} + ke_{x+Q}$ and $(e_x a)\bar{a} = e_x + k\bar{k}e_x = e_x + e_x = 0$.

Set $s = P + Q$, then one obtains

$$e_x^{t_a} = \begin{cases} ke_{x^{\sigma_s}} & , \quad x = P \\ \bar{k}e_{x^{\sigma_s}} & , \quad x = Q \\ \bar{k}e_{x^{\sigma_s}} & , \quad (P|x) = 0 \text{ and } (Q|x) = 1 \\ ke_{x^{\sigma_s}} & , \quad (P|x) = 1 \text{ and } (Q|x) = 1 \\ ke_{x^{\sigma_s}} & , \quad (P|x) = 1 \text{ and } (Q|x) = 0 \\ e_{x^{\sigma}} & , \quad \textit{otherwise.} \end{cases}$$

Hence, $\langle t_a | a \in \mathsf{a}_2 \rangle \cong M = \langle D_0, \hat{W} \rangle$ and as $\hat{W}$ normalizes D_0 by Proposition 2.1. Then $M = D_0 \rtimes \hat{W}$. □

Definition 2.2. For a M-set Δ corresponding to $s = s_\Delta$ and reflection $\sigma = \sigma_s = \sigma_{s_\Delta}$, we define $\sigma_\Delta(k)$ where $k^3 = 1$, by

$$e_x^{\sigma_\Delta} = \begin{cases} ke_{x_s^\sigma} & , \quad x \in \Delta \\ k^{-1}e_{x_s^\sigma} & , \quad x \in \Delta^* \\ e_x = e_{x_s^\sigma} & , \quad x \in \Delta_0 \end{cases}$$

Note 1. As $\sigma_{\Delta^*}(k) = \sigma_\Delta(k^{-1})$, we may use the notation $\sigma_\Delta(1) = \sigma_{\Delta^*}(1) = \sigma_s$. Hence, there are $3 \cdot 36 = 108$ such transpositions.

Theorem 2.6. *The set* $\{\sigma_\Delta(k), \Delta, k, k^3 = 1\} = \bigcup_{\substack{s \\ Q(s)=1}} \sigma_s^{D_0}$ *is a set of 108 3-transpositions in* M.

Proof. The $\bigcup_{\substack{s \\ Q(s)=1}} \sigma_s^{D_0}$ has order 108 by Theorem 2.3. The elements in M are of shape $\hat{\sigma}_1^d$, where $d \in D_0$, σ is a reflection in W. We have to show that $X = \bigcup_{\substack{s \\ Q(s)=1}} \hat{\sigma}_s^d$ is a set of 3-transpositions in M. So, let σ_1, σ_2 be two reflections in $W, d \in D_0$, then $(\hat{\sigma}_1^d)^{\hat{\sigma}_2} = \hat{\sigma}_1^{\hat{\sigma}_2 d^{\hat{\sigma}_2}} = (\widehat{\sigma_1\sigma_2})^{d^{\hat{\sigma}_2}} \in \cup \hat{\sigma}^{D_0}$. As M is transitive on X, it follows that $X^y = X$, $\forall y \in X$.

If $\sigma_1^{\sigma_2} = \sigma_1$, then $(\hat{\sigma}_1^d)^{\hat{\sigma}_2} = \hat{\sigma}_1^{d^{\hat{\sigma}_2}}$ has order 2.

If $(s_{\Delta_1}|s_{\Delta_2}) = 1$, then $o(\sigma_1\sigma_2) = 3$, $\langle \sigma_2, \sigma_3 \rangle \cong S_3$ and $(\hat{\sigma}_1^d)^{\hat{\sigma}_2} = (\hat{\sigma}_1^{\hat{\sigma}_2})^{d^{\hat{\sigma}_2}}$ is of order 3. □

3 Acknowledgment.

The authors thank the Public Authority for Applied Education and Training for supporting the research project no. BE-19-11.

References

[1] F. Alawam, M. Bani-Ata, A new existence proof of $F_{i_{22}}$ in ${}^2E_6(2)$, a computational approach, *Communications in Algebra*, doi.org/10.1080/00927872.2020.1722826.(to appear).

[2] S. Aldhafeeri, M. Bani-Ata, On construction of Lie algebras of type $E_6(K)$ for fields K of characteristic two, *Contributions to Algebra and Geometry*, **85,** (2017), 529–534.

[3] M. Bani-Ata, Quadratic forms on the 27-dimensional modules for E_6 in Characteristic two, *Journal of Lie Theory*, **33,** no. 1, (2022), 75–86.

[4] A. Galt, A. Staroletov, On splitting of the normalizer of maximal torus in $E_6(q^*)$, *Algebra Colloquium*, **26,** no. 2, (2019), 329–350.