International Journal of Mathematics and
Computer Science, **17**(2022), no. 3, 1375–1381

On $\alpha(\Lambda, sp)$-continuous multifunctions

Chokchai Viriyapong, Chawalit Boonpok

Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham, 44150, Thailand

email: chokchai.v@msu.ac.th, chawalit.b@msu.ac.th

(Received December 24, 2021, Accepted April 24, 2022)

Abstract

In this paper, we deal with the concept of $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, we investigate some characterizations of $\alpha(\Lambda, sp)$-continuous multifunctions.

1 Introduction

The concept of strongly semi-continuous functions was introduced by Noiri [8]. Mashhour et al. [5] called strongly semi-continuous functions α-continuous and obtained several properties of such functions. Noiri [7] investigated some characterizations of α-continuous functions and established the relationships between α-continuous functions and several known functions. Popa and Noiri [9] extended the concept of α-continuous functions to multifunctions and obtained several characterizations of α-continuous multifunctions. Abd El-Monsef et al. [1] introduced a weak form of open sets called β-open sets. This notion was also called semi-preopen sets in the sense of Andrijević [2]. Noiri and Hatir [6] introduced the notion of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [3], Boonpok introduced the concepts of (Λ, sp)-open sets and

Key words and phrases: $\alpha(\Lambda, sp)$-open set, $\alpha(\Lambda, sp)$-continuous multifunction.
AMS (MOS) Subject Classifications: 54C08, 54C60.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

(Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The purpose of the present paper is to introduce the notion of $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations of $\alpha(\Lambda, sp)$-continuous multifunctions are discussed.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [6] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a *Λ_{sp}-set* [6] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*.

Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda, sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda, sp)}$. A subset A of a topological space (X, τ) is said to be *$s(\Lambda, sp)$-open* (resp. *$\alpha(\Lambda, sp)$-open*) if $A \subseteq [A_{(\Lambda, sp)}]^{(\Lambda, sp)}$ (resp. $A \subseteq [[A_{(\Lambda, sp)}]^{(\Lambda, sp)}]_{(\Lambda, sp)}$) [3]. The complement of a $s(\Lambda, sp)$-open (resp. $\alpha(\Lambda, sp)$-open) set is said to be *$s(\Lambda, sp)$-closed* (resp. *$\alpha(\Lambda, sp)$-closed*). The family of all $s(\Lambda, sp)$-open (resp. $\alpha(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $\alpha\Lambda_{sp}O(X, \tau)$). The intersection of all $\alpha(\Lambda, sp)$-closed (resp. $s(\Lambda, sp)$-closed) sets containing A is called the *$\alpha(\Lambda, sp)$-closure* (resp. *$s(\Lambda, sp)$-closure*) of A and is denoted by $A^{\alpha(\Lambda, sp)}$ (resp. $A^{s(\Lambda, sp)}$). The union of all $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) sets contained in A is called the *$\alpha(\Lambda, sp)$-interior* (resp. *$s(\Lambda, sp)$-interior*) of A and is denoted by $A_{\alpha(\Lambda, sp)}$ (resp. $A_{s(\Lambda, sp)}$).

By a multifunction $F : (X, \tau) \to (Y, \sigma)$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively;

that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$ and for each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For any (Λ, sp)-open set V of a topological space (Y, σ), we denote $V^+ = \{B \in \mathcal{P}(Y) \mid B \subseteq V\}$ and $V^- = \{B \in \mathcal{P}(Y) \mid B \cap V \neq \emptyset\}$.

3 Characterizations

In this section, we introduce the notion of $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations of $\alpha(\Lambda, sp)$-continuous multifunctions are discussed.

Lemma 3.1. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \alpha\Lambda_{sp}O(X, \tau)$;

(2) $U \subseteq A \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ *for some (Λ, sp)-open set U;*

(3) $U \subseteq A \subseteq U^{s(\Lambda,sp)}$ *for some (Λ, sp)-open set U;*

(4) $A \subseteq [A_{(\Lambda,sp)}]^{s(\Lambda,sp)}$.

Lemma 3.2. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $[A^{(\Lambda,sp)}]_{s(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$;

(2) $A^{\alpha(\Lambda,sp)} = A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

Definition 3.3. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be $\alpha(\Lambda, sp)$-continuous at a point $x \in X$ if, for any (Λ, sp)-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, there exists $U \in \alpha\Lambda_{sp}O(X, \tau)$ containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be $\alpha(\Lambda, sp)$-continuous if F has this property at each point of X.*

Theorem 3.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *F is $\alpha(\Lambda, sp)$-continuous at a point $x \in X$;*

(2) *$x \in [[F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$ for any (Λ, sp)-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$.*

Proof. (1) ⇒ (2): Let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. There exists $U \in \alpha\Lambda_{sp}O(X, \tau)$ containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+(V_1) \cap F^-(V_2)$. Since $U \in \alpha\Lambda_{sp}O(X, \tau)$, we have

$$x \in U \subseteq [U_{(\Lambda,sp)}]^{s(\Lambda,sp)} \subseteq [[F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}.$$

(2) ⇒ (1): Let $x \in X$ and let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. Then, $x \in F^+(V_1) \cap F^-(V_2)$ and by (2), $x \in [[F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$. Thus,

$$F^+(V_1) \cap F^-(V_2) \subseteq [[F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}]^{s(\Lambda,sp)},$$

by Lemma 3.1, $F^+(V_1) \cap F^-(V_2) \in \alpha\Lambda_{sp}O(X, \tau)$. Put $U = F^+(V_1) \cap F^-(V_2)$, then U is an $\alpha(\Lambda, sp)$-open set of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. □

Let A be a subset of a topological space (X, τ). A subset B of X is said to be $\alpha(\Lambda, sp)$-*neighborhood* of A if there exists $U \in \alpha\Lambda_{sp}O(X, \tau)$ such that $A \subseteq U \subseteq B$. A subset B is called (Λ, sp)-*neighborhood which intersects* A if there exists an $\alpha(\Lambda, sp)$-open set V of X such that $V \subseteq B$ and $V \cap A \neq \emptyset$.

Theorem 3.5. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *F is $\alpha(\Lambda, sp)$-continuous;*

(2) *$F^+(V_1) \cap F^-(V_2) \in \alpha\Lambda_{sp}O(X, \tau)$ for any (Λ, sp)-open sets V_1, V_2 of Y;*

(3) *$F^-(K_1) \cup F^+(K_2)$ is (Λ, sp)-closed in X for any (Λ, sp)-closed sets K_1, K_2 of Y;*

(4) *$[[F^-(B_1) \cup F^+(B_2)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})$ for any subsets B_1, B_2 of Y;*

(5) *$[F^-(B_1) \cup F^+(B_2)]^{\alpha(\Lambda,sp)} \subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})$ for any subsets B_1, B_2 of Y;*

(6) *for each point $x \in X$, for each (Λ, sp)-neighborhood V_1 of $F(x)$ and for each (Λ, sp)-neighborhood V_2 which intersects $F(x)$, $F^+(V_1) \cap F^-(V_2)$ is an $\alpha(\Lambda, sp)$-neighborhood of x;*

(7) *for each point $x \in X$, for each (Λ, sp)-neighborhood V_1 of $F(x)$ and for each (Λ, sp)-neighborhood V_2 which intersects $F(x)$, there exists an $\alpha(\Lambda, sp)$-neighborhood U of x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$.*

Proof. (1) ⇒ (2): Let V_1, V_2 be any (Λ, sp)-open sets of Y and let

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then, $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, by Theorem 3.4,

$$x \in [[F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}.$$

Thus, $F^+(V_1) \cap F^-(V_2) \subseteq [[F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$. It follows from Lemma 3.1 that $F^+(V_1) \cap F^-(V_2) \in \alpha\Lambda_{sp}O(X, \tau)$.

(2) ⇒ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ and $F^-(Y - B) = X - F^+(B)$ for every subset B of Y.

(3) ⇒ (4): Let B_1, B_2 be any subsets of Y. Then, $B_1^{(\Lambda,sp)}$ and $B_2^{(\Lambda,sp)}$ are $(\Lambda, sp)-$closed in Y, by (3), $F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})$ is $\alpha(\Lambda, sp)$-open Y. Thus,

$$[[F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)}).$$

Since $F^-(B_1) \subseteq F^-(B_1^{(\Lambda,sp)})$ and $F^+(B_2) \subseteq F^+(B_2^{(\Lambda,sp)})$, we have

$$[[F^-(B_1) \cup F^+(B_2)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)}).$$

(4) ⇒ (5): By Lemma 3.2, we have $[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = [A^{(\Lambda,sp)}]_{s(\Lambda,sp)}$ for any subset A of X. It follows from Lemma 3.2 that

$$A^{\alpha(\Lambda,sp)} = A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} = A \cup [A^{(\Lambda,sp)}]_{s(\Lambda,sp)}.$$

This completes the proof of the implication.

(5) ⇒ (3): Let K_1, K_2 be any (Λ, sp)-closed sets of Y. By (5), we have $[F^-(K_1) \cup F^+(K_2)]^{\alpha(\Lambda,sp)} \subseteq F^-(K_1^{(\Lambda,sp)}) \cup F^+(K_2^{(\Lambda,sp)}) = F^-(K_1) \cup F^+(K_2)$ and hence $F^-(K_1) \cup F^+(K_2)$ is (Λ, sp)-closed in X.

(2) $\Rightarrow$ (6): Let $x \in X$ and let V_1 be a (Λ, sp)-neighborhood of $F(x)$. Let V_2 be a (Λ, sp)-neighborhood which intersects $F(x)$. There exist (Λ, sp)-open sets U_1 and U_2 such that $F(x) \subseteq U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $F(x) \cap U_2 \neq \emptyset$. Thus, $x \in F^+(U_1) \cap F^-(U_2) \subseteq F^+(V_1) \cap F^-(V_2)$. Since $F^+(U_1) \cap F^-(U_2)$ is $\alpha(\Lambda, sp)$-open, we have $F^+(U_1) \cap F^-(U_2)$ is an $\alpha(\Lambda, sp)$-neighborhood of x.

(6) $\Rightarrow$ (7): Let $x \in X$ and let V_1 be a (Λ, sp)-neighborhood of $F(x)$. Let V_2 be a (Λ, sp)-neighborhood which intersects $F(x)$. Put

$$U = F^+(V_1) \cap F^-(V_2),$$

then U is an $\alpha(\Lambda, sp)$-neighborhood of x, $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$.

(7) $\Rightarrow$ (1): Let $x \in X$ and let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. Then, V_1 is a (Λ, sp)-neighborhood of $F(x)$ and V_2 is a (Λ, sp)-neighborhood which intersects $F(x)$. There exists an $\alpha(\Lambda, sp)$-neighborhood U of x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Then, there exists $W \in \alpha\Lambda_{sp}O(X, \tau)$ such that $x \in W \subseteq U$; hence $F(W) \subseteq V_1$ and $F(w) \cap V_2 \neq \emptyset$ for every $w \in W$. This shows that F is $\alpha(\Lambda, sp)$-continuous. $\square$

Acknowledgment. This research project was partially supported by Mahasarakham University.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Assiut Univ., **12,** (1983), 77–90.

[2] D. Andrijević, On b-open sets, Mat. Vesnik, **48,** (1996), 59–64.

[3] C. Boonpok, (Λ, sp)-closed sets and related topics in topological spaces, WSEAS Tran. Math., **19,** (2020), 312–322.

[4] C. Berge, Espaces topologiques fonctions multivoques, Dunod, Paris, 1959.

[5] A. S. Mashhour, I. A. Hasanein, S. N. El- Deeb, α-continuous and α-open mappings, Acta Math. Hungar., **41,** (1983), 213–218.

[6] T. Noiri, E. Hatir, Λ_{sp}-sets and some weak separation axioms, Acta Math. Hungar., **103,** no. 3, (2004), 225–232.

[7] T. Noiri, On α-continuous functions, Časopis Pěst. Mat., **109,** (1984), 118–126.

[8] T. Noiri, A function which preserves connected spaces, Časopis Pěst. Mat., **107,** (1982), 393–396.

[9] V. Popa, T. Noiri, Characterizations of α-continuous multifunctions, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., **23,** no. 1, (1993), 29–38.