International Journal of Mathematics and
Computer Science, **17**(2022), no. 3, 1383–1391

The Lorentz transformations. A new approach

Jairo E. Castillo[1], **Jorge E. Pinzón**[2], **Alvaro H. Salas**[3]

[1]Universidad Distrital Francisco José de Caldas
Bogotá Colombia

[2]Universidad del Tolima
Ibague, Colombia

[3]Universidad Nacional de Colombia
sede Manizales, Colombia

email: jcastillo@udistrital.edu.co, jepinzon@ut.edu.co, ahsalass@unal.edu.co

(Received March 1, 2022, Accepted April 27, 2022)

Abstract

In a didactic way, we obtain the Lorentz's transformations without using the speed of light constancy postulate which is the most difficult to understand . The postulate of the constancy of the speed of light is obtained as a consequence of Eisntein's principle of relativity and the properties of space-time. We show that the Lorentz transformations, obtained from the properties of space-time and Eisten's principle of relativity, form a continuous group of symmetry of the system and as a direct consequence the postulate of the constancy of the velocity of the light.

1 Introduction

The special theory of relativity conceived by Albert Einstein [3] is considered one of the greatest achievements of the last century. It is the theory of space-time in the absence of gravity. The special theory of relativity is presented

Keywords and phrases: Lorentz's transformations, continuous group, symmetry.
AMS (MOS) Subject Classifications: 83C15, 83C20.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

in many engineering and science textbooks [1],[4],[5][6]. However, in these textbooks and many others, the Lorentz transformations are obtained from the postulates about the properties of space and time, Einstein's principle of relativity, and the postulate of the constancy of the speed of light, the latter being the most difficult to accept.

As is known, one of the most important postulates in the special theory of relativity is the homogeneity of space-time; that is, there is no privileged origin in physical terms; the experimental results are independent of the place and date in which it is carried out, as long as exactly the same initial conditions are preserved. In other words, having chosen a coordinate system in space-time and therefore an origin, one has the privilege of moving the origin to any other point without altering the description given to the physical system that is considered. The system is said to be invariant with respect to this transformation. It is clear that this transformation, displacement of the origin, is a continuous transformation; that is, the origin can be displaced in amounts as small as desired. In the mathematical sense, these transformations form a group. The situation is thus summarized, saying that the translations form a continuous group of transformations that leave the physical system invariant and it is said that the group is a continuous group of symmetry of the system.

On the one hand, the main objective of this work is to obtain the Lorentz transformations in a new didactic way that consists of not using the postulate of the constancy of the speed of light, which is the most difficult to understand.

2 Deduction of the Lorentz transformations without the Postulate of the Constancy of the speed of Light

The Lorentz transformations are deduced from the following postulates:

2.1 Postulate No. 1. Space is Homogenous

By homogeneity of space we mean the equivalence of all its parts: Every mechanical system located in a region of free space, taken at random, will move in the same way as anywhere else.

2.2 Postulate No. 2. Space is Isotrope

The isotropy of space is the equivalence in it of all its directions: the behavior of the mechanical system does not depend on the orientation.

2.3 Postulate No. 3. Time is Homogenous

The homogeneity of time means that all its moments are equivalent: the movement of a mechanical system will pass in the same way regardless of the moment of time that it was "started".

2.4 Postulate No. 4. Einstein Principle of Relativity

All the laws of nature are equivalent in all inertial frames of reference. This means that the equations that express the laws of nature must be invariant in relation to the transformations of coordinates and time when passing from one inertial reference system to another.

The most general form of the transformations of the coordinates and of the time when passing from an inertial frame of reference $\sum$ to another $\sum'$ that moves with a velocity v with respect to $\sum$ have the following aspect:

$$\begin{aligned} x' =& f_1(x, y, z, t) \\ y' =& f_2(x, y, z, t) \\ z' =& f_3(x, y, z, t) \\ t' =& f_4(x, y, z, t) \end{aligned} \tag{1}$$

Let's see the natural conditions to which the transformation functions f_1, f_2,f_3, f_4 are subject to. From the homogeneity of space and time it follows that the transformations must be linear:

$$x' = kx + ly + mz + nt + p, \text{ etc,} \tag{2}$$

Where k,l, m, n,p, ... are constant coefficients. If these magnitudes were functions of the coordinates and of time, it would mean that the transformation law (2) would be different for different points in space and different moments of time, which contradicts postulate No. 1. Of course, the coefficients k,l ,m, n, p, ... can depend on the relative velocity v.

To specify, the corresponding axes are parallel and the relative movement runs along the x axis at a speed v while the reference origins have been chosen so that for $t = 0$. The point $x' = y' = z' = 0$ (origin of system $\sum'$), coincides with the point $x = y = z = 0$ (origin of system $\sum$). See figure (1)

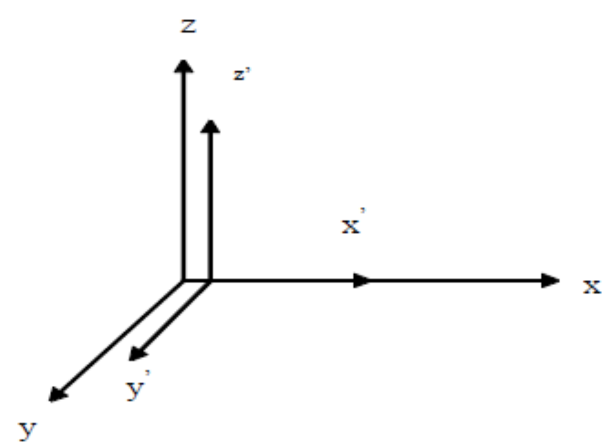

Figure 1

The hands of the clock in the system $\sum'$ have been set so that at the moment of the coincidence of the coordinate origins they show the time $t' = 0$. Under such conditions the independent terms in equality (2), the constant p, etc., are reduced to zero.

On the other hand, since the xy plane coincides with the $x'y'$ plane, it means that with $z' = 0$ we must have $z = 0$, with the particularity that these equalities have to be made with all values of x', y', t' and respectively of x, y, t. This is only possible if the link between z and z' looks like $z' = \lambda z$, where $\lambda = const$. Due to the isotropy of space, there must exist between y and y' the same bond with the same coefficient λ; $y' = \lambda y$. The transformations for x' and t' are written as follows:

$$\begin{aligned} x' &= kx + nt + ly + mz \\ t' &= \alpha x + \mu t + \sigma y + \delta z. \end{aligned} \tag{3}$$

In the plane $x' = 0$ we have $x = vt$ for all z and y, since the system $\sum'$ moves with respect to $\sum$ at speed v. Replacing these values of x and x' in the first equality of (3), we have $0 = k(vt) + nt$; $l = m = 0$ where $n = -kv$. In the system $\sum'$ the clock has been set so that for $x = 0$ and $t = 0$, also

$t' = 0$, which is only possible when $\delta = \sigma = 0$. As a result, the following formulas are obtained for the transformations (1).

$$\begin{aligned} x' =& kx - kvt = k(x - vt) \\ y' =& \lambda y \\ z' =& \lambda z \\ t' =& \mu t + \alpha x. \end{aligned} \tag{4}$$

The isotropy of space supposes a symmetry of space. Due to the symmetry of space, the transformation formulas (4) should not change if the signs of the x-axis direction and the sign of the speed of the system $\sum'$ change at the same time.

Namely:

$$\begin{aligned} -x' =& k(-v)(-x + vt) \\ y' =& \lambda(-v)y \\ z' =& \lambda(-v)z \\ t' =& \mu(-v)t - \alpha(-v)x \end{aligned} \tag{5}$$

Comparing (4) and (5) we have:

$$k(-v) = k(v),\ \alpha(-v) = -\alpha(v),\ \lambda(-v) = \lambda(v),\ \mu(-v) = \mu(v) \tag{6}$$

The procedure to derive the transformation formulas (4) from the properties of space and time is classical, and can be found in a good text on modern physics or classical electrodynamics such as [1].

Moving away from the classical procedure, a new function $\eta(v)$ is defined as follows [2]:

$$\eta(v) = -\frac{v\mu(v)}{\alpha(v)} \tag{7}$$

Note that $\eta(-v) = \eta(v)$ is a symmetric function.

Solving for α of (7) and replacing in the last equation of (4), we have that the transformation formulas for x and t are the following:

$$\begin{aligned} x' =& k(v)(x - vt) \ \ ,\ \ t' = \mu(t - \frac{v}{\eta}x) \\ x =& k(-v)(x' + vt') \ \ ,\ t = \mu(t' + \frac{v}{\eta}x') \end{aligned} \tag{8}$$

Multiplying x' by μ and t' by vk, and adding:

$$x'\mu + t'vk = \mu k(1 - \frac{v^2}{\eta})x. \tag{9}$$

Solving for x from (9)

$$x = \frac{x'\mu + t'vk}{\mu k(1 - \frac{v^2}{\eta})} = \frac{x'}{k(1 - \frac{v^2}{\eta})} + \frac{vt'}{\mu(1 - \frac{v^2}{\eta})} \tag{10}$$

Comparing (10) with $x = kx' + kvt'$ we have, by analogy in (4) the coordinate transformation of the system $\sum'$ to $\sum$ is given by:

$$x = kx' + kvt' \tag{11}$$

Comparing (10) and (11),

$$\begin{aligned} k &= \frac{1}{k(1 - \frac{v^2}{\eta})} \quad , \quad k = \frac{1}{\mu(1 - \frac{v^2}{\eta})} \quad \text{or} \\ k^2 &= \frac{1}{(1 - \frac{v^2}{\eta})} \quad , \quad k\mu = \frac{1}{(1 - \frac{v^2}{\eta})} \text{ from where } \mu = k \end{aligned} \tag{12}$$

From the third equation of (12)

$$k = \frac{1}{\sqrt{1 - \frac{v^2}{\eta}}} \tag{13}$$

Taking into account (13), the transformation law has the following form:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{\eta}}}, y' = \lambda y, z' = \lambda z, t' = \frac{t - \frac{v}{\eta}x}{\sqrt{1 - \frac{v^2}{\eta}}} \tag{14}$$

3 Determination of η

To determine the unknown function η, symmetric with respect to v, the principle of relativity is used (postulate No. 4).

The transformations (14) must be universal and are used to go from an inertial reference frame $\sum$ to another $\sum'$ and, of course, from $\sum'$ to another inertial reference frame $\sum''$.

That is, having chosen a coordinate system in space-time and therefore an origin, one has the privilege of moving the origin to any other point, without this altering in any way the description that is given of the physical system that is considered. The system is said to be invariant with respect to this transformation. It is clear that this transformation, displacement of the origin, is a continuous transformation; that is, the origin can be displaced in as small amounts as you want. In the mathematical sense, these transformations form a group. The situation is summarized by saying that the translations form a continuous group of transformations that leave the physical system invariant and the group is said to be a continuous group of symmetry of the system.

The transformation formulas for x' and t' of the system $\sum'$ with respect to $\sum$ are given by:

$$x' = k(v_1)(x - v_1 t) \quad , \quad t' = k(v_1)(t - \frac{v_1}{\eta(v_1)}x) \tag{15}$$

where v_1 is the velocity of the system $\sum'$ with respect to $\sum$, the transformation formulas of x'' and t'' of the system ($\sum''$) with respect to x' and t' of the system $\sum'$ are given by:

$$x'' = k(v_2)(x' - v_2 t') \quad , \quad t\,'' = k(v_2)(t' - \frac{v_2}{\eta(v_2)}x') \tag{16}$$

where v_2 is the velocity of the system $\sum''$ with respect to $\sum'$. Replacing x' in x'' we have:

$$x'' = (k(v_1)k(v_2) - \frac{k(v_1)k(v_2)v_1 v_2}{\eta(v_1)})x - (k(v_1)k(v_2)v_2 - k(v_1)k(v_2)v_1)t \tag{17}$$

Comparing (16) with $x'' = k(v_3)x - k(v_3)v_3 t$ where v_3 is the velocity of the system $\sum''$ with respect to $\sum$, we have

$$\begin{aligned} k(v_3) &= k(v_1)k(v_2) - \frac{k(v_1)k(v_2)v_1 v_2}{\eta(v_1)} \\ k(v_3)v_3 &= (k(v_1)k(v_2)v_2 - k(v_1)k(v_2)v_1) \end{aligned} \tag{18}$$

In the same way, replacing t' in t'', we have:

$$t'' = (k(v_1)k(v_2) - \frac{k(v_1)k(v_2)v_1 v_2}{\eta(v_2)})t - (\frac{k(v_1)k(v_2)v_1}{\eta(v_1)} - \frac{k(v_1)k(v_2)v_2}{\eta(v_2)})x \tag{19}$$

Comparing (19) with $t'' = k(v_3)t - \frac{k(v_3)v_3}{\eta(v_{3)}}x$ we have:

$$\begin{aligned} k(v_3) &= k(v_1)k(v_2) - \frac{k(v_1)k(v_2)v_1v_2}{\eta(v_2)} \\ \frac{k(v_3)v_3}{\eta(v_{3)}} &= \frac{k(v_1)k(v_2)v_1}{\eta(v_1)} - \frac{k(v_1)k(v_2)v_2}{\eta(v_2)} \end{aligned} \tag{20}$$

From the first equation in (18) and the first equation in (20), we get a very interesting result

$$k(v_1)k(v_2) - \frac{k(v_1)k(v_2)v_1v_2}{\eta(v_1)} = k(v_1)k(v_2) - \frac{k(v_1)k(v_2)v_1v_2}{\eta(v_2)} \tag{21}$$

which is possible if and only if

$$\eta(v_1) = \eta(v_2) = Cons = c \tag{22}$$

where c is the constant of the speed of light in vacuum.

The constant η in the transformation formulas (14) has a dimension of L^2T^{-2} and it is the experiment that gives us its numerical value.

Finally, to determine λ in the transformation formulas (14), postulate No. 2 (the isotropy of space) is used because the directions are equivalent $\lambda(v) = \lambda(-v)$. Using the transformation from y to y' , after y' to y we have $y = \lambda^2 y$ that is $\lambda^2 = 1$,

$\lambda = \pm 1$. The value of $\lambda = -1$ responds to the inverse orientation of the y and y' axes, so that only for the value $\lambda = 1$ will there be correspondence with figure (1).

From what has been said above, we conclude that the postulate of the constancy of the speed of light can be obtained as a consequence of Einstein's principle of relativity and the properties of space-time.

4 Conclusions

Einstein's principle of relativity is a logical extension of Galileo's principle of relativity with which the student does not see any logical contradiction. However, the principle of the constancy of the speed of light is in direct contradiction with classical physics since it contradicts the relative sum of speeds, is very anti-intuitive and difficult to accept.

From equation (22), we infer that there is a maximum speed $\eta(v_1) = \eta(v_2) = Cons$ and it is the same in all inertial reference systems which leads to the postulate of the constancy of the speed of light. From this point of view, the postulate of the constancy of the speed of light is a consequence of Einstein's postulate of relativity and of the properties of space and time.

The above shows the possibility of presenting the Lorentz transformations in a didactic way without the postulate of the constancy of the speed of light.

In the transformations (14) the numerical value of the constant η is obtained from the experimental results.

The postulate of the constancy of the speed of light can be stated as a corollary. The proposed methodology can be of great help for students who are starting in the study of modern physics and the special theory of relativity.

5 Acknowledgment

The authors would like to thank the University Francisco Jose de Caldas for the support to carry out this work.

References

[1] M. Bredov, V. Rumiantsev, I. Toptiquin, Classical Electrodynamics, Ed Mir, Moscow, 1985, 23.

[2] J. P. Terletsky. Paradoxes of the Special Theory of Relativity. Moscow, 1965, 24–25.

[3] A. Einstein, Ann. Phys., **17,** (1905),891–921.

[4] R. Serway, J. Jewett, Physics for Scientists and Engineers, Seventh Edition, 2008.

[5] F. W. Sears, M. W. Zemansky, H. D. Young. College Physics, Seventh Edition, 1990.

[6] J. D. Jackson, Classical Electrodynamics, 3rd. ed., 1998.

[7] A. A. Logunov, Theory of relativity and gravitation, Ed Nauka, Moscow, 1987.