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(Λ, sp)-continuous multifunctions

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Abstract

This paper deals with the concept of (Λ, sp)-continuous multifunctions. Moreover, we investigate some characterizations of (Λ, sp)-continuous multifunctions.

1 Introduction

The concepts of openness and continuity are fundamental to the investigation of general topology. Weaker and stronger forms of open sets play an important role in topological spaces. By using these sets, many authors introduced and studied various types of generalizations of continuity. In 1983, Abd El-Monsef et al. [1] introduced a weak form of open sets called β-open sets which was also called semi-preopen sets in the sense of Andrijević [2]. Noiri and Hatir [5] introduced the notion of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets. In [3], the author introduced the concepts of (Λ, sp)-closed sets and (Λ, sp)-open sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The purpose of the present paper is to introduce the notion of (Λ, sp)-continuous multifunctions. Moreover, several characterizations of (Λ, sp)-continuous multifunctions are discussed.

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2 Preliminaries

Throughout this paper and unless explicitly stated, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed. Let A be a subset of a topological space (X,τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X,τ) is called *β-open* [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X,τ) is denoted by $\beta(X,\tau)$. A subset $\Lambda_{sp}(A)$ [5] is defined as follows:

$$\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X,\tau)\}.$$

If $A = \Lambda_{sp}(A)$, then A is called a *Λ_{sp}-set* [5]. A subset A of a topological space (X,τ) is called *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. The family of all (Λ, sp)-open sets in a topological spaces (X,τ) is denoted by $\Lambda_{sp}O(X,\tau)$. Let A be a subset of a topological space (X,τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda,sp)}$. By a multifunction $F : (X,\tau) \to (Y,\sigma)$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively. That is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$ and for each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For any (Λ, sp)-open set V of a topological space (Y,σ), we use the notations $V^+ = \{B \in \mathcal{P}(Y) \mid B \subseteq V\}$ and $V^- = \{B \in \mathcal{P}(Y) \mid B \cap V \neq \emptyset\}$.

3 (Λ, sp)-continuous multifunctions

We begin this section by introducing the notion of (Λ, sp)-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$.*

Theorem 3.2. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is (Λ, sp)-continuous;

(2) $x \in [F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}$ for any (Λ, sp)-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$;

(3) $F^+(V_1) \cap F^-(V_2)$ is (Λ, sp)-open in X for any (Λ, sp)-open sets V_1, V_2 of Y;

(4) $F^-(K_1) \cup F^+(K_2)$ is (Λ, sp)-closed in X for any (Λ, sp)-closed sets K_1, K_2 of Y;

(5) $[F^-(B_1) \cup F^+(B_2)]^{(\Lambda,sp)} \subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})$ for any subsets B_1, B_2 of Y;

(6) $F^-([B_1]_{(\Lambda,sp)}) \cap F^+([B_2]_{(\Lambda,sp)}) \subseteq [F^-(B_1) \cap F^+(B_2)]_{(\Lambda,sp)}$ for any subsets B_1, B_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. There exists a (Λ, sp)-open set U containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Thus, $U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $x \in [F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}$.

(2) $\Rightarrow$ (3): Let V_1, V_2 be any (Λ, sp)-open sets of Y and let

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then, $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By (2), $x \in [F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}$ and hence $F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}$. This shows that $F^+(V_1) \cap F^-(V_2)$ is (Λ, sp)-open in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(4) $\Rightarrow$ (5): Let B_1, B_2 be any subsets of Y. Then, $B_1^{(\Lambda,sp)}$ and $B_2^{(\Lambda,sp)}$ are (Λ, sp)-closed in Y, by (4),

$$\begin{aligned} [F^-(B_1) \cup F^+(B_2)]^{(\Lambda,sp)} &\subseteq [F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})]^{(\Lambda,sp)} \\ &= F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)}). \end{aligned}$$

(5) ⇒ (6): Let B_1, B_2 be any subsets of Y. By (5), we have

$$\begin{aligned} X - [F^-(B_1) \cap F^+(B_2)]_{(\Lambda,sp)} &= [(X - F^-(B_1)) \cup (X - F^+(B_2))]^{(\Lambda,sp)} \\ &= [F^+(Y - B_1) \cup F^-(Y - B_2)]^{(\Lambda,sp)} \\ &\subseteq F^+([Y - B_1]^{(\Lambda,sp)}) \cup F^-([Y - B_2]^{(\Lambda,sp)}) \\ &= F^+(Y - [B_1]_{(\Lambda,sp)}) \cup F^-(Y - [B_2]_{(\Lambda,sp)}) \\ &= X - [F^-([B_1]_{(\Lambda,sp)}) \cap F^+([B_2]_{(\Lambda,sp)})] \end{aligned}$$

and hence $F^-([B_1]_{(\Lambda,sp)}) \cap F^+([B_2]_{(\Lambda,sp)}) \subseteq [F^-(B_1) \cap F^+(B_2)]_{(\Lambda,sp)}$.

(6) ⇒ (1): Let $x \in X$ and let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, by (6),

$$F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1) \cap F^-(V_2)]_{(\Lambda,sp)}.$$

Now, put $U = F^+(V_1) \cap F^-(V_2)$, then U is a (Λ, sp)-open set of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for each $z \in U$. This shows that F is (Λ, sp)-continuous. □

Definition 3.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open sets V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $f(U) \subseteq V$.*

Corollary 3.4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is (Λ, sp)-continuous;

(2) $x \in [f^{-1}(V)]_{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y containing $f(x)$;

(3) $f^{-1}(V)$ is (Λ, sp)-open in X for every (Λ, sp)-open set V of Y;

(4) $f^{-1}(K)$ is (Λ, sp)-closed in X for every (Λ, sp)-closed set K of Y;

(5) $[f^{-1}(B)]^{(\Lambda,sp)} \subseteq f^{-1}(B^{(\Lambda,sp)})$ for every subset B of Y;

(6) $f^{-1}(B_{(\Lambda,sp)}) \subseteq [f^{-1}(B)]_{(\Lambda,sp)}$ for every subset B of Y.

Definition 3.5. [3] *Let A be a subset of a topological space (X, τ). The $\theta(\Lambda, sp)$-closure of A, $A^{\theta(\Lambda,sp)}$, is defined as follows:*

$$A^{\theta(\Lambda,sp)} = \{x \in X \mid A \cap\ U^{(\Lambda,sp)} \neq \emptyset \text{ for each } U \in \Lambda_{sp}O(X, \tau) \text{ containing } x\}.$$

Definition 3.6. [3] *A subset A of a topological space (X, τ) is called $\theta(\Lambda, sp)$-closed if $A = A^{\theta(\Lambda, sp)}$. The complement of a $\theta(\Lambda, sp)$-closed set is said to be $\theta(\Lambda, sp)$-open.*

Definition 3.7. [3] *A topological space (X, τ) is said to be Λ_{sp}-regular if, for each (Λ, sp)-closed set F and each $x \notin F$, there exist disjoint (Λ, sp)-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 3.8. [3] *Let (X, τ) be a Λ_{sp}-regular space. Then, the following properties hold:*

(1) $A^{(\Lambda, sp)} = A^{\theta(\Lambda, sp)}$ for every subset A of X.

(2) Every (Λ, sp)-open set is $\theta(\Lambda, sp)$-open.

Theorem 3.9. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is Λ_{sp}-regular, the following properties are equivalent:*

(1) F is (Λ, sp)-continuous;

(2) $F^{-}(B_1^{\theta(\Lambda, sp)}) \cup F^{+}(B_2^{\theta(\Lambda, sp)})$ is (Λ, sp)-closed in X for any subsets B_1, B_2 of Y;

(3) $F^{-}(K_1) \cup F^{+}(K_2)$ is (Λ, sp)-closed in X for any $\theta(\Lambda, sp)$-closed sets K_1, K_2 of Y;

(4) $F^{-}(V_1) \cap F^{+}(V_2)$ is (Λ, sp)-open in X for any $\theta(\Lambda, sp)$-open sets V_1, V_2 of Y.

Proof. (1) $\Rightarrow$ (2): Let B_1, B_2 be any subsets of Y. By Lemma 3.8, $B_1^{\theta(\Lambda, sp)}$ and $B_2^{\theta(\Lambda, sp)}$ are (Λ, sp)-closed in Y, by Theorem 3.2, $F^{-}(B_1^{\theta(\Lambda, sp)}) \cup F^{+}(B_2^{\theta(\Lambda, sp)})$ is (Λ, sp)-closed in X.

(2) $\Rightarrow$ (3): Let K_1, K_2 be any $\theta(\Lambda, sp)$-closed sets of Y. Then, $K_1^{\theta(\Lambda, sp)} = K_1$ and $K_2^{\theta(\Lambda, sp)} = K_2$, by (2), $F^{-}(K_1) \cup F^{+}(K_2)$ is (Λ, sp)-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^{+}(Y - B) = X - F^{-}(B)$ and $F^{-}(Y - B) = X - F^{+}(B)$ for any subset B of Y.

(4) $\Rightarrow$ (1): Let V_1, V_2 be any (Λ, sp)-open sets of Y. Since (Y, σ) is Λ_{sp}-regular, V_1, V_2 are $\theta(\Lambda, sp)$-open in Y and by (4), $F^{-}(V_1) \cap F^{+}(V_2)$ is (Λ, sp)-open in X. Thus, F is (Λ, sp)-continuous by Theorem 3.2. □

Corollary 3.10. *For a function $f : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is Λ_{sp}-regular, the following properties are equivalent:*

(1) f *is* (Λ, sp)*-continuous;*

(2) $f^{-1}(B^{\theta(\Lambda,sp)})$ *is* (Λ, sp)*-closed in* X *for every subset* B *of* Y*;*

(3) $f^{-1}(K)$ *is* (Λ, sp)*-closed in* X *for every* $\theta(\Lambda, sp)$*-closed set* K *of* Y*;*

(4) $f^{-1}(V)$ *is* (Λ, sp)*-open in* X *for every* $\theta(\Lambda, sp)$*-open set* V *of* Y*.*

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References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Assiut Univ., **12,** (1983), 77–90.

[2] D. Andrijević, On b-open sets, Mat. Vesnik, **48,** (1996), 59–64.

[3] C. Boonpok, (Λ, sp)-closed sets and related topics in topological spaces, WSEAS Tran. Math., **19,** (2020), 312–322.

[4] C. Berge, Espaces topologiques fonctions multivoques, Dunod, Paris, 1959.

[5] T. Noiri, E. Hatir, Λ_{sp}-sets and some weak separation axioms, Acta Math. Hungar., **103,** no. 3, (2004), 225–232.