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Upper and lower $\beta(\Lambda, sp)$-continuous multifunctions

Jeeranunt Khampakdee, Chawalit Boonpok

Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham, 44150, Thailand

email: jeeranunt.k@msu.ac.th, chawalit.b@msu.ac.th

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Abstract

Our main purpose is to introduce the concepts of upper and lower $\beta(\Lambda, sp)$-continuous multifunctions. In particular, we investigate some characterizations of upper and lower $\beta(\Lambda, sp)$-continuous multifunctions.

1 Introduction

In 1983, Abd El-Monsef et al. [1] defined β-continuity in topological spaces as a generalization of semi-continuity [4]. Popa and Noiri [6] extended the concept of β-continuous functions to multifunctions and investigated the notions of upper and lower β-continuous multifunctions. In 2004, Noiri and Hatir [5] introduced the notion of Λ_{sp}-sets in terms of β-open sets [1] and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. Boonpok [2] introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The purpose of the present paper is to introduce the notions of upper and lower $\beta(\Lambda, sp)$-continuous multifunctions. Moreover, we discuss some characterizations of upper and lower $\beta(\Lambda, sp)$-continuous multifunctions.

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2 Preliminaries

Throughout this paper, unless explicitly stated, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed. Let A be a subset of a topological space (X,τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X,τ) is said to be *β-open* [1] if $A\subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X,τ) is denoted by $\beta(X,\tau)$. A subset $\Lambda_{sp}(A)$ [5] is defined as follows:

$$\Lambda_{sp}(A)=\cap\{U \mid A\subseteq U, U\in\beta(X,\tau)\}.$$

If $A=\Lambda_{sp}(A)$, then A is called a *Λ_{sp}-set* [5]. A subset A of a topological space (X,τ) is called *(Λ,sp)-closed* [2] if $A=T\cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ,sp)-closed set is called *(Λ,sp)-open*. Let A be a subset of a topological space (X,τ). A point $x\in X$ is called a *(Λ,sp)-cluster point* [2] of A if $A\cap U\neq\emptyset$ for every (Λ,sp)-open set U of X containing x. The set of all (Λ,sp)-cluster points of A is called the *(Λ,sp)-closure* [2] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ,sp)-open sets contained in A is called the *(Λ,sp)-interior* [2] of A and is denoted by $A_{(\Lambda,sp)}$. A subset A of a topological space (X,τ) is said to be *$\beta(\Lambda,sp)$-open* (resp. *$s(\Lambda,sp)$-open*) [2] if $A\subseteq[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A\subseteq[A_{(\Lambda,sp)}]^{(\Lambda,sp)}$). The complement of a $\beta(\Lambda,sp)$-open set is said to be *$\beta(\Lambda,sp)$-closed*. The intersection of all $\beta(\Lambda,sp)$-closed sets containing A is called the *$\beta(\Lambda,sp)$-closure* of A and is denoted by $A^{\beta(\Lambda,sp)}$. The union of all $\beta(\Lambda,sp)$-open sets contained in A is called the *$\beta(\Lambda,sp)$-interior* of A and is denoted by $A_{\beta(\Lambda,sp)}$.

Following [3], by a multifunction $F:(X,\tau)\to(Y,\sigma)$ we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively; that is, $F^+(B)=\{x\in X \mid F(x)\subseteq B\}$ and

$$F^-(B)=\{x\in X \mid F(x)\cap B\neq\emptyset\}.$$

In particular, $F^-(y)=\{x\in X \mid y\in F(x)\}$ for each point $y\in Y$ and for each $A\subseteq X$, $F(A)=\cup_{x\in A}F(x)$.

3 Upper and lower $\beta(\Lambda,sp)$-continuous multifunctions

In this section, we introduce the notions of upper and lower $\beta(\Lambda,sp)$-continuous multifunctions. Moreover, we discuss several characterizations of upper and lower $\beta(\Lambda, sp)$-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) upper $\beta(\Lambda, sp)$-continuous at a point $x \in X$ if, for each (Λ, sp)-open set V of Y containing $F(x)$, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V$;

(ii) lower $\beta(\Lambda, sp)$-continuous at a point $x \in X$ if, for each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $U \subseteq F^-(V)$;

(iii) upper (lower) $\beta(\Lambda, sp)$-continuous if F has this property at each point $x \in X$.

Definition 3.2. [2] *A subset N of a topological space (X, τ) is said to be Λ_{sp}-nowhere dense if $[N^{(\Lambda,sp)}]_{(\Lambda,sp)} = \emptyset$.*

Theorem 3.3. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper $\beta(\Lambda, sp)$-continuous at $x \in X$;

(2) for each (Λ, sp)-open set U of X containing x and each (Λ, sp)-open set V of Y with $x \in F^+(V)$, $F^+(V) \cap U$ is not Λ_{sp}-nowhere dense;

(3) for each (Λ, sp)-open set U of X containing x and each (Λ, sp)-open set V of Y with $x \in F^+(V)$, there exists a (Λ, sp)-open set G of X such that $\emptyset \neq G \subseteq U$ and $G \subseteq [F^+(V)]^{(\Lambda,sp)}$;

(4) for each (Λ, sp)-open set V of Y with $x \in F^+(V)$, there exists a $s(\Lambda, sp)$-open set U of X containing x such that $U \subseteq [F^+(V)]^{(\Lambda,sp)}$;

(5) $x \in [[[F^+(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y with $x \in F^+(V)$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (4): Let V be any (Λ, sp)-open set of Y containing $F(x)$. By $\mathcal{U}(x)$ we denote the family of all (Λ, sp)-open sets of X containing x. For each $U \in \mathcal{U}(x)$, there exists a (Λ, sp)-open set G_U of X such that $\emptyset \neq G_U \subseteq U$ and $G_U \subseteq [F^+(V)]^{(\Lambda,sp)}$. Put $W = \cup\{G_U \mid U \in \mathcal{U}(x)\}$. Then, W is (Λ, sp)-open set of X, $x \in W^{(\Lambda,sp)}$ and $W \subseteq [F^+(V)]^{(\Lambda,sp)}$. Moreover, we put $U_0 = W \cup \{x\}$. Then, $W \subseteq U_0 \subseteq W^{(\Lambda,sp)}$. Thus, U_0 is a $s(\Lambda, sp)$-open set of X containing x and $U_0 \subseteq [F^+(V)]^{(\Lambda,sp)}$.

(4) ⇒ (5): Let V be any (Λ, sp)-open set of Y containing $F(x)$. There exists a $s(\Lambda, sp)$-open set U of X containing x such that $U \subseteq [F^+(V)]^{(\Lambda,sp)}$. Thus, $x \in U \subseteq [U_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [[[F^+(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

(5) ⇒ (1): The proof is obvious. □

Theorem 3.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower $\beta(\Lambda, sp)$-continuous at $x \in X$;

(2) for each (Λ, sp)-open set U of X containing x and each (Λ, sp)-open set V of Y with $x \in F^-(V)$, $F^-(V) \cap U$ is not Λ_{sp}-nowhere dense;

(3) for each (Λ, sp)-open set U of X containing x and each (Λ, sp)-open set V of Y with $x \in F^-(V)$, there exists a (Λ, sp)-open set G of X such that $\emptyset \neq G \subseteq U$ and $G \subseteq [F^-(V)]^{(\Lambda,sp)}$;

(4) for each (Λ, sp)-open set V of Y with $x \in F^-(V)$, there exists a $s(\Lambda, sp)$-open set U of X containing x such that $U \subseteq [F^-(V)]^{(\Lambda,sp)}$;

(5) $x \in [[[F^-(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y with $x \in F^-(V)$.

Proof. The proof is similar to that of Theorem 3.3. □

Lemma 3.5. *Let A be a subset of a topological space (X, τ). Then, $A^{\beta(\Lambda,sp)} = A \cup [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$.*

Theorem 3.6. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper $\beta(\Lambda, sp)$-continuous;

(2) $F^+(V)$ is $\beta(\Lambda, sp)$-open in X for every (Λ, sp)-open set V of Y;

(3) $F^-(K)$ is $\beta(\Lambda, sp)$-closed in X for every (Λ, sp)-closed set K of Y;

(4) $[F^-(B)]^{\beta(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $[[[F^-(B)]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any (Λ, sp)-open set of Y and let $x \in F^+(V)$. Then, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V$. Therefore, $x \in U \subseteq [[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq [[[F^+(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, $F^+(V) \subseteq [[[F^+(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $F^+(V)$ is $\beta(\Lambda, sp)$-open in X.

(2) $\Rightarrow$ (3): This follows immediately from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4): For any subset B of Y, $B^{(\Lambda,sp)}$ is (Λ, sp)-closed in Y, by (3), $F^-(B^{(\Lambda,sp)})$ is $\beta(\Lambda, sp)$-closed in X and hence $[F^-(B)]^{\beta(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4) and Lemma 3.5,

$$[[[F^-(B)]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [F^-(B)]^{\beta(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)}).$$

(5) $\Rightarrow$ (2): Let V be any (Λ, sp)-open set of Y. Then, $Y - V$ is (Λ, sp)-closed in Y, by (5), $X - F^+(V) = F^-(Y - V) \supseteq [[[F^-(Y - V)]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = [[[X - F^+(V)]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} = X - [[[F^+(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Thus, $F^+(V) \subseteq [[[F^+(V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence $F^+(V)$ is $\beta(\Lambda, sp)$-open in X.

(2) $\Rightarrow$ (1): Let $x \in X$ and let V be any (Λ, sp)-open set of Y containing $F(x)$. Then, $x \in F^+(V)$, by (2), $F^+(V)$ is $\beta(\Lambda, sp)$-open in X. Put $U = F^+(V)$. Then, U is a $\beta(\Lambda, sp)$-open set of X containing x and $F(U) \subseteq V$. This shows that F is upper $\beta(\Lambda, sp)$-continuous. □

Theorem 3.7. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower $\beta(\Lambda, sp)$-continuous;

(2) $F^-(V)$ is $\beta(\Lambda, sp)$-open in X for every (Λ, sp)-open set V of Y;

(3) $F^+(K)$ is $\beta(\Lambda, sp)$-closed in X for every (Λ, sp)-closed set K of Y;

(4) $[F^+(B)]^{\beta(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $[[[F^+(B)]_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(6) $F([[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$ for every subset A of X;

(7) $F(A^{\beta(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$ for every subset A of X.

Proof. The proofs are similar to the proof of Theorem 3.6 that the properties (1), (2), (3), (4) and (5) are equivalent. We shall only prove the following implications:

(5) $\Rightarrow$ (6): Let A be any subset of X. By (5), we have $[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [[[F^+(F(A))]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^+([F(A)]^{(\Lambda,sp)})$ and hence

$$F([[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}.$$

(6) $\Rightarrow$ (7): Let A be any subset of X. By (6) and Lemma 3.5, $F(A^{\beta(\Lambda,sp)}) = F(A \cup [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}) = F(A) \cup F([[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$.

(7) $\Rightarrow$ (3): Let K be any (Λ, sp)-closed set of Y. By (7), we have $F([F^+(K)]^{\beta(\Lambda,sp)}) \subseteq [F(F^+(K))]^{(\Lambda,sp)} \subseteq K^{(\Lambda,sp)} = K$. Thus, $[F^+(K)]^{\beta(\Lambda,sp)} \subseteq F^+(K)$. This shows that $F^+(K)$ is $\beta(\Lambda, sp)$-closed in X. □

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