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Characterizations of almost quasi (Λ, sp)-continuous multifunctions

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Abstract

In this paper, we deal with the concept of almost quasi (Λ, sp)-continuous multifunctions. In particular, we investigate some characterizations of almost quasi (Λ, sp)-continuous multifunctions.

1 Introduction

The concept of quasi continuous functions was first introduced by Marcus [5]. Popa [8] introduced and studied the notion of almost quasi continuous functions. Bânzaru and Crivăţ [3] introduced and investigated the notion of quasi continuous multifunctions. Popa and Noiri [7] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. Noiri and Hatir [6] introduced the notions of Λ_{sp}-closed sets and spg-closed sets and investigated some properties of Λ_{sp}-closed sets and spg-closed sets. By considering the notion of Λ_{sp}-sets, Boonpok [2] introduced and investigated (Λ, sp)-closed sets, (Λ, sp)-open sets and (Λ, sp)-closure operators. The purpose of the present paper is to introduce the notion of almost quasi (Λ, sp)-continuous multifunctions. Moreover, we

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discuss several characterizations and some basic properties of almost quasi (Λ, sp)-continuous multifunctions.

2 Preliminaries

Throughout this paper, unless explicitly stated, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [6] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. If $A = \Lambda_{sp}(A)$, then A is called a *Λ_{sp}-set* [6]. A subset A of a topological space (X, τ) is called *(Λ, sp)-closed* [2] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [2] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [2] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [2] of A and is denoted by $A_{(\Lambda,sp)}$. A subset A of a topological space (X, τ) is said to be *$s(\Lambda, sp)$-open* (resp. *$p(\Lambda, sp)$-open*, *$\beta(\Lambda, sp)$-open*, *$r(\Lambda, sp)$-open*) if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$, $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$) [2]. The complement of a $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) set is said to be *$s(\Lambda, sp)$-closed* (resp. *$p(\Lambda, sp)$-closed*, *$\beta(\Lambda, sp)$-closed*, *$r(\Lambda, sp)$-closed*). The intersection of all $s(\Lambda, sp)$-closed sets containing A is called the *$s(\Lambda, sp)$-closure* of A and is denoted by $A^{s(\Lambda,sp)}$. The union of all $s(\Lambda, sp)$-open sets contained in A is called the *$s(\Lambda, sp)$-interior* of A and is denoted by $A_{s(\Lambda,sp)}$. By a multifunction $F : (X, \tau) \to (Y, \sigma)$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$ and for each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For any (Λ, sp)-open set V of a topological space (Y, σ), we denote $V^+ = \{B \in \mathcal{P}(Y) \mid B \subseteq V\}$ and $V^- = \{B \in \mathcal{P}(Y) \mid B \cap V \neq \emptyset\}$.

3 Characterizations

We begin this section by introducing the concept of almost quasi (Λ, sp)-continuous multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau) \to (Y, \sigma,)$ is said to be almost quasi (Λ, sp)-continuous at a point $x \in X$ if for any (Λ, sp)-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$ and each open set U containing x, there exists a nonempty (Λ, sp)-open set G of X such that $G \subseteq U$, $F(G) \subseteq V_1^{s(\Lambda,sp)}$ and $F(z) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in G$. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be almost quasi (Λ, sp)-continuous if F is almost quasi (Λ, sp)-continuous at each point of X.*

Lemma 3.2. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $A^{s(\Lambda,sp)} = A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$.

(2) $A_{s(\Lambda,sp)} = A \cap [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

Theorem 3.3. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is almost quasi (Λ, sp)-continuous;

(2) for each $x \in X$ and every (Λ, sp)-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a $s(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V_1^{s(\Lambda,sp)}$ and $F(z) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in U$;

(3) $F^+(V_1) \cap F^-(V_2)$ is $s(\Lambda, sp)$-open in X for every $r(\Lambda, sp)$-open sets V_1, V_2 of Y;

(4) $F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{s(\Lambda,sp)}$ for every (Λ, sp)-open sets V_1, V_2 of Y;

(5)

$$[F^-([[B_1^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[B_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)})]^{s(\Lambda,sp)}$$
$$\subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})$$

for every subsets B_1, B_2 of Y;

(6) $F^+(V_1) \cap F^-(V_2) \subseteq [[F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ *for every* (Λ, sp)*-open sets* V_1, V_2 *of* Y.

Proof. (1) ⇒ (2): Let $\mathcal{U}(x)$ the family of all (Λ, sp)-open sets of X containing x. Let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. For each $H \in \mathcal{U}(x)$, there exists a nonempty (Λ, sp)-open set G_H such that $G_H \subseteq H$, $F(G_H) \subseteq V_1^{s(\Lambda,sp)}$ and $F(y) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $y \in G_H$. Let $W = \cup\{G_H \mid H \in \mathcal{U}(x)\}$. Then, W is (Λ, sp)-open in X, $x \in W^{(\Lambda,sp)}$, $F(W) \subseteq V_1^{s(\Lambda,sp)}$ and $F(w) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq W^{(\Lambda,sp)}$. Thus, U is a $s(\Lambda, sp)$-open set of X containing x such that $F(U) \subseteq V_1^{s(\Lambda,sp)}$ and $F(z) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in U$.

(2) ⇒ (3): Let V_1, V_2 be any $r(\Lambda, sp)$-open sets of Y and let

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then, $F(x) \in V_1^+ \cap V_2^-$ and there exists a $s(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Therefore, $x \in U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $x \in [F^+(V_1) \cap F^-(V_2)_{s(\Lambda,sp)}$. Thus, $F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1) \cap F^-(V_2)]_{s(\Lambda,sp)}$. This shows that $F^+(V_1) \cap F^-(V_2)$ is $s(\Lambda, sp)$-open in X.

(3) ⇒ (4): Let V_1, V_2 be any (Λ, sp)-open sets of Y such that

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then, $F(x) \subseteq V_1 \subseteq V_1^{s(\Lambda,sp)}$ and $\emptyset \neq F(x) \cap V_2 \subseteq F(x) \cap V_2^{s(\Lambda,sp)}$. Thus, $x \in F^+(V_1^{s(\Lambda,sp)})$ and $x \in F^-(V_2^{s(\Lambda,sp)})$. By Lemma 3.2, $V_1^{s(\Lambda,sp)}$ and $V_2^{s(\Lambda,sp)}$ are $r(\Lambda, sp)$-open sets and by (3), $F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})$ is $s(\Lambda, sp)$-open in X and $x \in [F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{s(\Lambda,sp)}$. Consequently, we obtain $F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{s(\Lambda,sp)}$.

(4) ⇒ (5): Let B_1, B_2 be any subsets of Y. Then, $Y - B_1^{(\Lambda,sp)}$ and $Y - B_2^{(\Lambda,sp)}$ are (Λ, sp)-open sets of Y. By (4) and Lemma 3.2,

$$\begin{aligned}
&X - (F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)})) \\
&= F^+(Y - B_1^{(\Lambda,sp)}) \cap F^-(Y - B_2^{(\Lambda,sp)}) \\
&\subseteq [F^+([Y - B_1^{(\Lambda,sp)}]^{s(\Lambda,sp)}) \cap F^-([Y - B_2^{(\Lambda,sp)}]^{s(\Lambda,sp)})]_{s(\Lambda,sp)} \\
&= X - [F^-([[B_1^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[B_2^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{s(\Lambda,sp)}
\end{aligned}$$

and hence

$$\begin{aligned}
&[F^-([[B_1^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[B_2^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{s(\Lambda,sp)} \\
&\subseteq F^-(B_1^{(\Lambda,sp)}) \cup F^+(B_2^{(\Lambda,sp)}).
\end{aligned}$$

$(5) \Rightarrow (6)$: Let V_1, V_2 be any (Λ, sp)-open sets of Y. Then, $Y - V_1$ and $Y - V_2$ are (Λ, sp)-closed sets of Y. By (5) and Lemma 3.2, we have

$$\begin{aligned}&[[F^-([[Y - V_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[Y - V_2]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &\subseteq F^-(Y - V_1) \cup F^+(Y - V_2) = X - (F^+(V_1) \cap F^-(V_2)).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&[[F^-([[Y - V_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[Y - V_2]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &= [[F^-(Y - [V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+(Y - [V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &= [[(X - F^+(V_1^{s(\Lambda,sp)})) \cup (X - F^-(V_2^{s(\Lambda,sp)}))]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &= X - [[F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}.\end{aligned}$$

Thus, $F^+(V_1) \cap F^-(V_2) \subseteq [[F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

$(6) \Rightarrow (1)$: Let $x \in X$ and let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By (6), we have

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq [[F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)},$$

by Lemma 3.2, $x \in F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{s(\Lambda,sp)}$. Put $U = [F^+(V_1^{s(\Lambda,sp)}) \cap F^-(V_2^{s(\Lambda,sp)})]_{s(\Lambda,sp)}$, then U is an $s(\Lambda, sp)$-open set of X containing x such that $F(U) \subseteq V_1^{s(\Lambda,sp)}$ and $F(z) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in U$. This shows that F is almost quasi (Λ, sp)-continuous. □

Theorem 3.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is almost quasi (Λ, sp)-continuous;

(2) $[F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ for every $\beta(\Lambda, sp)$-open sets V_1, V_2 of Y;

(3) $[F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ for every $s(\Lambda, sp)$-open sets V_1, V_2 of Y;

(4) $F^+(V_1) \cap F^-(V_2) \subseteq [F^+([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cap F^-([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{s(\Lambda,sp)}$ for every $p(\Lambda, sp)$-open sets V_1, V_2 of Y.

Proof. The proof follows from Theorem 3.3. □

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