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Non-null distribution of Kullback's statistic for testing equality of correlation matrices

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Abstract

In this article, we derive the asymptotic non-null distribution of Kullback's statistic L^* for testing equality of several correlation matrices. First, the distribution of a function of L^* is obtained in series involving cumulative distribution function of a standard normal distribution by inverting the characteristic function from which the distribution of L^* is deduced.

1 Introduction

If the covariance matrix of α-th population is given by Σ_α and Δ_α is a diagonal matrix of the standard deviations for the population α, then $P_\alpha = \Delta_\alpha^{-1}\Sigma_\alpha\Delta_\alpha^{-1}$ is the correlation matrix

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for the population α. The null hypothesis that all k populations have the same correlation matrices may be stated as $H : P_1 = \cdots = P_k$.

Let the vectors $\boldsymbol{x}_{\alpha 1}, \boldsymbol{x}_{\alpha 2}, \ldots, \boldsymbol{x}_{\alpha N_\alpha}$ be a random sample of size $N_\alpha = n_\alpha + 1$ for $\alpha = 1, 2, \ldots, k$ from k independent multivariate populations of dimensionality p. Further, let $\bar{\boldsymbol{x}}_\alpha = \sum_{i=1}^{N_\alpha} \boldsymbol{x}_{\alpha i}/N_\alpha$, $A_\alpha = \sum_{i=1}^{N_\alpha} (\boldsymbol{x}_{\alpha i} - \bar{\boldsymbol{x}}_\alpha)(\boldsymbol{x}_{\alpha i} - \bar{\boldsymbol{x}}_\alpha)'$ and $S_\alpha = A_\alpha/N_\alpha$. Let D_α be a diagonal matrix of the square roots of the diagonal elements of S_α. The sample correlation matrix R_α is then given by $R_\alpha = D_\alpha^{-1} S_\alpha D_\alpha^{-1}$. Let $n = \sum_{\alpha=1}^{k} n_\alpha$ and $\bar{R} = \sum_{\alpha=1}^{k} n_\alpha R_\alpha / n$.

Kullback [18] derived the statistic $L^* = \sum_{\alpha=1}^{k} n_\alpha \ln\{\det(\bar{R})/\det(R_\alpha)\}$ for testing the equality of k correlation matrices based on independent samples from multivariate populations. This statistic was later examined by Jennrich [12] who observed that the statistic proposed by Kullback failed to have chi-square distribution ascribed to it. Gupta, Johnson and Nagar [10] have shown that although the Kullback's statistic L^* is not equal to the modified likelihood ration criterion, it may be considered an approximation to the modified likelihood ratio statistic when sampling from multivariate normal populations. They also derived asymptotic null distribution of L^* in series involving independent chi-square variables by expanding L^* in terms of other random variables and then inverting the expansion term by term.

For further insight the reader is referred to Aitkin [1], Aitkin Nelson [2], Ali, Fraser and Lee [3], Atiany and Sharif [5, 6], Browne [7, 8], Konishi [14], Konishi and Takakazu [17], Modarres and Jeriningan [19], and Schott [20].

In this article, we derive the asymptotic non-null distribution of the Kullback's statistic. In Section 2, we give several definitions and results used in the development of the non-null distribution. In sections 3 and 4, we obtain the asymptotic non-null distribution function of some function of L^* in terms of standard normal distribution function.

2 Preliminaries

The cumulants of order m are functions of the moments of order m or lower. Thus if the m^{th} order moment is finite, so is the m^{th} order cumulant. Let $\mu_i = E(X_i)$, $\mu_{ij} = E(X_i X_j)$, $\mu_{ijk} = E(X_i X_j X_k)$, and $\mu_{ijk\ell} = E(X_i X_j X_k X_\ell)$ and κ_i, κ_{ij}, κ_{ijk}, and $\kappa_{ijk\ell}$ be the corresponding cumulants. Then, Kaplan [13] and Boik [9] give the following relationship:

$$\kappa_i = \mu_i, \tag{2.1}$$

$$\kappa_{ij} = \mu_{ij} - \mu_i \mu_j, \tag{2.2}$$

$$\kappa_{ijk} = \mu_{ijk} - (\mu_i \mu_{jk} + \mu_j \mu_{ik} + \mu_k \mu_{ij}) + 2\mu_i \mu_j \mu_k, \tag{2.3}$$

$$\kappa_{ijk\ell} = \mu_{ijk\ell} - \sum^{4} \mu_i \mu_{jk\ell} - \sum^{3} \mu_{ij} \mu_{k\ell} + 2 \sum^{6} \mu_i \mu_j \mu_{k\ell} - 6 \mu_i \mu_j \mu_k \mu_\ell, \tag{2.4}$$

where the summations are over the possible ways of grouping the subscripts, and the number of terms resulting is written over the summation sign.

Define the random matrix $V_\alpha = (v_{\alpha ij})$ as

$$V_\alpha = \sqrt{n_\alpha}\left(\frac{1}{n_\alpha}\Delta_\alpha^{-1}A_\alpha\Delta_\alpha^{-1} - P_\alpha\right). \tag{2.5}$$

Then, the random matrices $V_\alpha^{(0)} = (v_{\alpha ij}^{(0)})$, $V_\alpha^{(1)} = (v_{\alpha ij}^{(1)})$ and $V_\alpha^{(2)} = (v_{\alpha ij}^{(2)})$ are defined as

$$V_\alpha^{(0)} = \operatorname{diag}(v_{\alpha 11}, v_{\alpha 22}, \ldots, v_{\alpha pp}), \tag{2.6}$$

$$V_\alpha^{(1)} = V_\alpha - \frac{1}{2}V_\alpha^{(0)}P_\alpha - \frac{1}{2}P_\alpha V_\alpha^{(0)}, \tag{2.7}$$

and

$$V_\alpha^{(2)} = \frac{1}{4}V_\alpha^{(0)}P_\alpha V_\alpha^{(0)} - \frac{1}{2}V_\alpha V_\alpha^{(0)} - \frac{1}{2}V_\alpha^{(0)}V_\alpha + \frac{3}{8}(V_\alpha^{(0)})^2P_\alpha + \frac{3}{8}P_\alpha(V_\alpha^{(0)})^2. \tag{2.8}$$

Konishi [15, 16] has shown that

$$R_\alpha = P_\alpha + \frac{1}{\sqrt{n_\alpha}}V_\alpha^{(1)} + \frac{1}{n_\alpha}V_\alpha^{(2)} + O_p(n_\alpha^{-3/2}).$$

The pooled estimate of the common correlation matrix is therefore

$$\bar{R} = \sum_{\alpha=1}^{k}\omega_\alpha R_\alpha = \bar{P} + \frac{1}{\sqrt{n}}\bar{V}^{(1)} + \frac{1}{n}\bar{V}^{(2)} + O_p(n^{-3/2}),$$

where $\omega_\alpha = n_\alpha/n$, $\bar{P} = \sum_{\alpha=1}^{k}\omega_\alpha P_\alpha$, $\bar{V}^{(1)} = \sum_{\alpha=1}^{k}\sqrt{\omega_\alpha}\,V_\alpha^{(1)}$ and $\bar{V}^{(2)} = \sum_{\alpha=1}^{k}V_\alpha^{(2)}$. The limiting distribution of $V_\alpha = \sqrt{n_\alpha}\,(\Delta_\alpha^{-1}A_\alpha\Delta_\alpha^{-1}/n_\alpha - P_\alpha)$ is normal with means 0 and covariances that depend on the fourth order cumulants of the parent population.

Gupta, Johnson and Nagar [10] have shown that

$$E(v_{\alpha ij}) = 0,$$

$$E(v_{\alpha ij}v_{\alpha k\ell}) = \kappa_{\alpha ijk\ell} + \rho_{\alpha ik}\rho_{\alpha j\ell} + \rho_{\alpha i\ell}\rho_{\alpha jk} + O(n_\alpha^{-1}),$$

and

$$E(v_{\alpha ij}v_{\alpha k\ell}v_{\alpha ab}) = \frac{1}{\sqrt{n_\alpha}}\Bigg[\kappa_{\alpha ijk\ell ab} + \sum^{12}\kappa_{\alpha ijka}\rho_{\alpha\ell b} + \sum^{4}\kappa_{\alpha ika}\kappa_{\alpha j\ell b} + \sum^{8}\rho_{\alpha ik}\rho_{\alpha ja}\rho_{\alpha\ell b}\Bigg] + O(n_\alpha^{-3/2}).$$

The random matrices $V_\alpha^{(0)}$, $V_\alpha^{(1)}$ and $V_\alpha^{(2)}$ are defined in (2.6), (2.7), and (2.8), respectively. The expectations associated with these random matrices are given as

$$E(v_{\alpha ij}^{(1)}) = 0, \tag{2.9}$$

$$\begin{aligned} E(v_{\alpha ij}^{(2)}) &= \frac{1}{4}\rho_{\alpha ij}\kappa_{\alpha iijj} - \frac{1}{2}(\kappa_{\alpha iiij} + \kappa_{\alpha ijjj}) + \frac{3}{8}\rho_{\alpha ij}(\kappa_{\alpha iiii} + \kappa_{\alpha jjjj}) \\ &\quad + \frac{1}{2}(\rho_{\alpha ij}^3 - \rho_{\alpha ij}) + O(n_\alpha^{-1}), \end{aligned} \tag{2.10}$$

$$\begin{aligned} E(v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)}) &= \kappa_{\alpha ijk\ell} - \frac{1}{2}(\rho_{\alpha ij}\kappa_{\alpha iik\ell} + \rho_{\alpha ij}\kappa_{\alpha jjk\ell} + \rho_{\alpha k\ell}\kappa_{\alpha ijkk} + \rho_{\alpha k\ell}\kappa_{\alpha ij\ell\ell}) \\ &\quad + \frac{1}{4}\rho_{\alpha ij}\rho_{\alpha k\ell}(\kappa_{\alpha iikk} + \kappa_{\alpha ii\ell\ell} + \kappa_{\alpha jjkk} + \kappa_{\alpha jj\ell\ell}) \\ &\quad - (\rho_{\alpha k\ell}\rho_{\alpha ik}\rho_{\alpha jk} + \rho_{\alpha k\ell}\rho_{\alpha i\ell}\rho_{\alpha j\ell} + \rho_{\alpha ij}\rho_{\alpha ik}\rho_{\alpha i\ell} + \rho_{\alpha ij}\rho_{\alpha jk}\rho_{\alpha j\ell}) \\ &\quad + \frac{1}{2}\rho_{\alpha ij}\rho_{\alpha k\ell}(\rho_{\alpha ik}^2 + \rho_{\alpha i\ell}^2 + \rho_{\alpha jk}^2 + \rho_{\alpha j\ell}^2) \\ &\quad + (\rho_{\alpha ik}\rho_{\alpha j\ell} + \rho_{\alpha i\ell}\rho_{\alpha jk}) + O(n_\alpha^{-1}), \end{aligned} \tag{2.11}$$

and

$$E(v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} v_{\alpha ab}^{(1)}) = \frac{1}{\sqrt{n_\alpha}}\left(t_{\alpha 1} - \frac{1}{2}t_{\alpha 2} + \frac{1}{4}t_{\alpha 3} - \frac{1}{8}t_{\alpha 4}\right) + O(n_\alpha^{-3/2}) \tag{2.12}$$

where

$$t_{\alpha 1} = \kappa_{\alpha ijk\ell ab} + \sum^{12}\kappa_{\alpha ijka}\kappa_{\alpha \ell b} + \sum^{4}\kappa_{\alpha ika}\kappa_{\alpha j\ell b} + \sum^{8}\rho_{\alpha ik}\rho_{\alpha ja}\rho_{\alpha \ell b}, \tag{2.13}$$

$$\begin{aligned} t_{\alpha 2} = \sum^{3}\rho_{\alpha ij}\Bigg[&\kappa_{\alpha iik\ell ab} + \kappa_{\alpha jjk\ell ab} + \sum^{12}(\kappa_{\alpha iika} + \kappa_{\alpha jjka})\rho_{\alpha \ell b} \\ &+ \sum^{4}(\kappa_{\alpha ika}\kappa_{\alpha i\ell b} + \kappa_{\alpha jka}\kappa_{\alpha j\ell b}) + \sum^{8}(\rho_{\alpha ik}\rho_{\alpha ia} + \rho_{\alpha jk}\rho_{\alpha ja})\rho_{\alpha \ell b}\Bigg], \end{aligned} \tag{2.14}$$

$$\begin{aligned} t_{\alpha 3} = \sum^{3}\rho_{\alpha ij}\rho_{\alpha k\ell}\Bigg[&\kappa_{\alpha iikkab} + \kappa_{\alpha ii\ell\ell ab} + \kappa_{\alpha jjkkab} + \kappa_{\alpha jj\ell\ell ab} \\ &+ \sum^{12}(\kappa_{\alpha iika}\rho_{\alpha kb} + \kappa_{\alpha ii\ell a}\rho_{\alpha \ell b} + \kappa_{\alpha jjka}\rho_{\alpha kb} + \kappa_{\alpha jj\ell a}\rho_{\alpha \ell b}) \\ &+ \sum^{4}(\kappa_{\alpha ika}\kappa_{\alpha ikb} + \kappa_{\alpha i\ell a}\kappa_{\alpha i\ell b} + \kappa_{\alpha jka}\kappa_{\alpha jkb} + \kappa_{\alpha j\ell a}\kappa_{\alpha j\ell b}) \\ &+ \sum^{8}(\rho_{\alpha ik}\rho_{\alpha ia}\rho_{\alpha bk} + \rho_{\alpha i\ell}\rho_{\alpha ia}\rho_{\alpha \ell b} + \rho_{\alpha jk}\rho_{\alpha ja}\rho_{\alpha bk} + \rho_{\alpha j\ell}\rho_{\alpha ja}\rho_{\alpha \ell b})\Bigg] \end{aligned} \tag{2.15}$$

and

$$t_{\alpha 4}=\rho_{\alpha ij}\rho_{\alpha k\ell}\rho_{\alpha ab}\sum^{8}\left[\kappa_{\alpha iikkaa}+\sum^{12}\kappa_{\alpha iika}\rho_{\alpha ka}+\sum^{4}\kappa^2_{\alpha ika}+\sum^{8}\rho_{\alpha ik}\rho_{\alpha ia}\rho_{\alpha ka}\right] \tag{2.16}$$

where the summations are over the possible ways of grouping the subscripts, and the number of terms resulting is written over the summation sign.

Finally, we give the following result from Gupta, Johnson and Nagar [10, Corollary 5.1.2].

Lemma 2.1. *Let the k sample correlation coefficients $r_1, r_2, \ldots, r_k$ be based on samples of sizes $N_1, N_2, \ldots, N_k$ from bivariate populations which are elliptically contoured with a common curtosis of 3κ and common correlation coefficient ρ. Then*

$$L^* = \left[(1-\rho^2)^2+(1+2\rho^2)\kappa\right]\frac{1+\rho^2}{(1-\rho^2)^2}\chi^2_{(k-1)}+O_p(n^{-1/2}).$$

3 Asymptotic moments

Define

$$G(R_1,\ldots,R_k)=\sqrt{n}[g(R_1,\ldots,R_k)-g(P_1,\ldots,P_k)],$$

where $g(R_1,\ldots,R_k)=\ln[\det(\bar{R})]-\sum_{\alpha=1}^k\omega_\alpha\ln[\det(R_\alpha)]$ and $g(P_1,\ldots,P_k)=\ln[\det(\bar{P})]-\sum_{\alpha=1}^k\omega_\alpha\ln[\det(P_\alpha)]$. Using $L^*=ng(R_1\ldots,R_k)$, it may be observed that

$$G(R_1,\ldots,R_k)=\frac{1}{\sqrt{n}}[L^*-ng(P_1,\ldots,P_k)]. \tag{3.1}$$

The purpose of this section is to derive expected values of $G(R_1,\ldots,R_k)$ and its functions.

First, we recall the following theorem proved in Gupta, Johnson and Nagar [10].

Theorem 3.1. *The expression $G=G(R_1,\ldots,R_k)$ may be written as*

$$G=G(R_1,\ldots,R_k)=T_1+\frac{1}{\sqrt{n}}T_2+O_p(n^{-1}),$$

where

$$T_1=\sum_{\alpha=1}^k\sum_{i<j}\sqrt{\omega_\alpha}\,\bar{h}_{\alpha ij}v^{(1)}_{\alpha ij}, \tag{3.2}$$

$$\begin{aligned}T_2=&\sum_{\alpha=1}^k\sum_{i<j}\bar{h}_{\alpha ij}v^{(2)}_{\alpha ij}+\sum_{\alpha=1}^k\sum_{i<j}\sum_{k<\ell}q_\alpha(ij,k\ell)v^{(1)}_{\alpha ij}v^{(1)}_{\alpha k\ell}\\&-\sum_{\alpha=1}^k\sum_{\beta=1}^k\sum_{i<j}\sum_{k<\ell}\sqrt{\omega_\alpha\omega_\beta}\,q(ij,k\ell)v^{(1)}_{\alpha ij}v^{(1)}_{\beta k\ell},\end{aligned} \tag{3.3}$$

$P_\alpha^{-1} = (\rho_\alpha^{ij})$, $\bar{P}^{-1} = (\bar{\rho}^{ij})$, $\bar{h}_{\alpha ij} = 2(\bar{\rho}^{ij} - \rho_\alpha^{ij})$, $q_\alpha(ij, k\ell) = \rho_\alpha^{i\ell}\rho_\alpha^{jk} + \rho_\alpha^{ik}\rho_\alpha^{j\ell}$ *and* $q(ij, k\ell) = \bar{\rho}^{i\ell}\bar{\rho}^{jk} + \bar{\rho}^{ik}\bar{\rho}^{j\ell}$.

Using results given in (2.9)–(2.12), one can easily observe that $E(T_1) = 0$, $E(T_1T_2) = O(n^{-1/2})$, $E(T_1^2T_2) = O(n^{-1})$, and $E(T_1T_2^2) = O(n^{-3/2})$.

Lemma 3.2. $E(G) = \frac{1}{\sqrt{n}}u_1 + O(n^{-1})$ *where*

$$\begin{aligned}
u_1 = \sum_{\alpha=1}^{k}\sum_{i<j}\sum_{k<\ell}&\{q_\alpha(ij,k\ell) - \omega_\alpha q(ij,k\ell)\}\Big[\kappa_{\alpha ijk\ell} \\
&- \frac{1}{2}(\rho_{\alpha ij}\kappa_{\alpha iik\ell} + \rho_{\alpha ij}\kappa_{\alpha jjk\ell} + \rho_{\alpha k\ell}\kappa_{\alpha ijkk} + \rho_{\alpha k\ell}\kappa_{\alpha ij\ell\ell}) \\
&+ \frac{1}{4}\rho_{\alpha ij}\rho_{\alpha k\ell}(\kappa_{\alpha iikk} + \kappa_{\alpha ii\ell\ell} + \kappa_{\alpha jjkk} + \kappa_{\alpha jj\ell\ell}) \\
&- (\rho_{\alpha k\ell}\rho_{\alpha ik}\rho_{\alpha jk} + \rho_{\alpha k\ell}\rho_{\alpha i\ell}\rho_{\alpha j\ell} + \rho_{\alpha ij}\rho_{\alpha ik}\rho_{\alpha i\ell} + \rho_{\alpha ij}\rho_{\alpha jk}\rho_{\alpha j\ell}) \\
&+ \frac{1}{2}\rho_{\alpha ij}\rho_{\alpha k\ell}(\rho_{\alpha ik}^2 + \rho_{\alpha i\ell}^2 + \rho_{\alpha jk}^2 + \rho_{\alpha j\ell}^2) + (\rho_{\alpha ik}\rho_{\alpha j\ell} + \rho_{\alpha i\ell}\rho_{\alpha jk})\Big] \\
&+ \sum_{\alpha=1}^{k}\sum_{i<j}\bar{h}_{\alpha ij}\Big[\frac{1}{4}\rho_{\alpha ij}\kappa_{\alpha iijj} - \frac{1}{2}(\kappa_{\alpha iiij} + \kappa_{\alpha ijjj}) + \frac{3}{8}\rho_{\alpha ij}(\kappa_{\alpha iiii} + \kappa_{\alpha jjjj}) \\
&+ \frac{1}{2}(\rho_{\alpha ij}^3 - \rho_{\alpha ij})\Big].
\end{aligned}$$

Proof. By using the expansion of G, the expected value of G is obtained as

$$E(G) = E\left[T_1 + \frac{1}{\sqrt{n}}T_2 + O_p(n^{-1})\right] = \frac{1}{\sqrt{n}}E\left(T_2\right) + O_p(n^{-1}),$$

where the second step has been obtained by observing that $E(T_1) = 0$. Now, taking expectation of (3.3), we obtain

$$\begin{aligned}
E(T_2) = &\sum_{\alpha=1}^{k}\sum_{i<j}\bar{h}_{\alpha ij}E(v_{\alpha ij}^{(2)}) + \sum_{\alpha=1}^{k}\sum_{i<j}\sum_{k<\ell}q_\alpha(ij,k\ell)E(v_{\alpha ij}^{(1)}v_{\alpha k\ell}^{(1)}) \\
&- \sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\sum_{i<j}\sum_{k<\ell}\sqrt{\omega_\alpha\omega_\beta}\,q(ij,k\ell)E(v_{\alpha ij}^{(1)}v_{\beta k\ell}^{(1)}).
\end{aligned}$$

The desired result is now obtained by using (2.10), (2.11) and noting that $E(v_{\alpha ij}^{(1)}v_{\beta k\ell}^{(1)}) = 0$ for $\alpha \neq \beta$. □

Corollary 3.3. *If samples are taken from bivariate populations, then*

$$u_1 = \sum_{\alpha=1}^{k} \left(\frac{1+\rho_\alpha^2}{(1-\rho_\alpha^2)^2} - \omega_\alpha \frac{1+\bar{\rho}^2}{(1-\bar{\rho}^2)^2} \right) \Bigg[(1-\rho_\alpha^2)^2 + \frac{1}{4}\rho_\alpha^2(\kappa_{\alpha1111} + \kappa_{\alpha2222}) + \left(1 + \frac{1}{2}\rho_\alpha^2\right)\kappa_{\alpha1122} - \rho_\alpha(\kappa_{\alpha1112} + \kappa_{\alpha1222}) \Bigg] + 2\sum_{\alpha=1}^{k} \left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2} \right) \Bigg[\frac{1}{4}\rho_\alpha\kappa_{\alpha1122} - \frac{1}{2}(\kappa_{\alpha1112} + \kappa_{\alpha1222}) + \frac{3}{8}\rho_\alpha(\kappa_{\alpha1111} + \kappa_{\alpha2222}) - \frac{1}{2}\rho_\alpha(1-\rho_\alpha^2) \Bigg].$$

Lemma 3.4. $E(G^2) = u_2 + O(n^{-1})$, *where*

$$u_2 = \sum_{\alpha=1}^{k} \sum_{i<j} \sum_{k<\ell} \omega_\alpha \bar{h}_{\alpha ij} \bar{h}_{\alpha k\ell} \Bigg[\kappa_{\alpha ijk\ell} - \frac{1}{2}(\rho_{\alpha ij}\kappa_{\alpha iik\ell} + \rho_{\alpha ij}\kappa_{\alpha jjk\ell} + \rho_{\alpha k\ell}\kappa_{\alpha ijkk} + \rho_{\alpha k\ell}\kappa_{\alpha ij\ell\ell}) + \frac{1}{4}\rho_{\alpha ij}\rho_{\alpha k\ell}(\kappa_{\alpha iikk} + \kappa_{\alpha ii\ell\ell} + \kappa_{\alpha jjkk} + \kappa_{\alpha jj\ell\ell}) - (\rho_{\alpha k\ell}\rho_{\alpha ik}\rho_{\alpha jk} + \rho_{\alpha k\ell}\rho_{\alpha i\ell}\rho_{\alpha j\ell} + \rho_{\alpha ij}\rho_{\alpha ik}\rho_{\alpha i\ell} + \rho_{\alpha ij}\rho_{\alpha jk}\rho_{\alpha j\ell}) + \frac{1}{2}\rho_{\alpha ij}\rho_{\alpha k\ell}(\rho_{\alpha ik}^2 + \rho_{\alpha i\ell}^2 + \rho_{\alpha jk}^2 + \rho_{\alpha j\ell}^2) + (\rho_{\alpha ik}\rho_{\alpha j\ell} + \rho_{\alpha i\ell}\rho_{\alpha jk}) \Bigg].$$

Proof. Using the expansion of G, one obtains

$$E(G^2) = E\left(T_1^2 + \frac{2}{\sqrt{n}}T_1T_2 + \frac{2}{n}T_2^2\right) + O(n^{-1}) = E(T_1^2) + O(n^{-1}).$$

Now, from (3.2), we have

$$E(T_1^2) = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \sum_{i<j} \sum_{k<\ell} \sqrt{\omega_\alpha\omega_\beta}\, \bar{h}_{\alpha ij}\, \bar{h}_{\beta k\ell} E(v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)}).$$

Finally, by noting that $E(v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)}) = 0$ for $\alpha \neq \beta$ and using (2.11), the desired result is obtained. □

Corollary 3.5. *If samples are taken from bivariate populations, then*

$$u_2 = 4\sum_{\alpha=1}^{k} \omega_\alpha \left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2} \right)^2 \Bigg[(1-\rho_\alpha^2)^2 + \frac{1}{4}\rho_\alpha^2(\kappa_{\alpha1111} + \kappa_{\alpha2222}) + \left(1 + \frac{1}{2}\rho_\alpha^2\right)\kappa_{\alpha1122} - \rho_\alpha(\kappa_{\alpha1112} + \kappa_{\alpha1222}) \Bigg].$$

Lemma 3.6. *$E(G^3) = \frac{1}{\sqrt{n}} u_3 + O(n^{-3/2})$ with*

$$u_3 = \sum_{\alpha=1}^{k} \sum_{i<j} \sum_{k<\ell} \sum_{a<b} \omega_\alpha \bar{h}_{\alpha ij} \bar{h}_{\alpha k\ell} \bar{h}_{\alpha ab} \left(t_{\alpha 1} - \frac{1}{2} t_{\alpha 2} + \frac{1}{4} t_{\alpha 3} - \frac{1}{8} t_{\alpha 4} \right) + O(n^{-3/2}),$$

where t_1, t_2, t_3 and t_4 are given by (2.13), (2.14), (2.15) and (2.16), respectively.

Proof. Using the expansion of G, one obtains

$$\begin{aligned} E(G^3) &= E\left(T_1^3 + \frac{3}{\sqrt{n}} T_1^2 T_2 + \frac{3}{n} T_1 T_2^2 + \frac{3}{n^{3/2}} T_2^3 \right) + O(n^{-3/2}) \\ &= E(T_1^3) + O(n^{-3/2}). \end{aligned}$$

Now, from (3.2), we have

$$E(T_1^3) = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \sum_{\gamma=1}^{k} \sum_{i<j} \sum_{k<\ell} \sum_{a<b} \sqrt{\omega_\alpha \omega_\beta \omega_\gamma}\, \bar{h}_{\alpha ij}\, \bar{h}_{\beta k\ell}\, \bar{h}_{\gamma ab} E(v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)} v_{\gamma ab}^{(1)})$$

and since, for $\alpha \neq \beta$ or $\alpha \neq \gamma$ or $\beta \neq \gamma$, $E(v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)} v_{\gamma ab}^{(1)}) = 0$, the above expression reduces to

$$E(T_1^3) = \sum_{\alpha=1}^{k} \sum_{i<j} \sum_{k<\ell} \sum_{a<b} \omega_\alpha \sqrt{\omega_\alpha}\, \bar{h}_{\alpha ij}\, \bar{h}_{\alpha k\ell}\, \bar{h}_{\alpha ab} E(v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} v_{\alpha ab}^{(1)}).$$

The final result is obtained by substituting for $E(v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} v_{\alpha ab}^{(1)})$ from (2.12). □

Corollary 3.7. *If samples are taken from bivariate populations, then*

$$u_3 = 8 \sum_{\alpha=1}^{k} \omega_\alpha \left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2} \right)^3 (k_\alpha^6 + k_\alpha^4 + k_\alpha^3),$$

where

$$\begin{aligned} k_\alpha^6 &= -\frac{1}{8} \rho_\alpha^3 (\kappa_{\alpha 111111} + \kappa_{\alpha 222222}) + \frac{3}{4} \rho_\alpha^2 (\kappa_{\alpha 111112} + \kappa_{\alpha 122222}) \\ &\quad - \frac{3}{2} \rho_\alpha \Big(\frac{1}{4} \rho_\alpha^2 + 1 \Big) (\kappa_{\alpha 111122} + \kappa_{\alpha 112222}) + \Big(\frac{3}{2} \rho_\alpha^2 + 1 \Big) \kappa_{\alpha 111222}, \\ k_\alpha^4 &= -\frac{3}{2} \rho_\alpha (1-\rho_\alpha^2)(\kappa_{\alpha 1111} + \kappa_{\alpha 2222}) + 3(1-\rho_\alpha^4)(\kappa_{\alpha 1112} + \kappa_{\alpha 1222}) \\ &\quad - 9\rho_\alpha (1-\rho_\alpha^2) \kappa_{\alpha 1122}, \\ k_\alpha^3 &= -\frac{1}{2} \rho_\alpha^3 (\kappa_{\alpha 111}^2 + \kappa_{\alpha 222}^2) + 3\rho_\alpha^2 (\kappa_{\alpha 111} \kappa_{\alpha 112} + \kappa_{\alpha 122} \kappa_{\alpha 222}) \\ &\quad - 3\rho_\alpha (\kappa_{\alpha 111} \kappa_{\alpha 122} + \kappa_{\alpha 112} \kappa_{\alpha 222}) - \frac{3}{2} \rho_\alpha^3 (\kappa_{\alpha 112}^2 + \kappa_{\alpha 122}^2) \\ &\quad - 3\rho_\alpha (\kappa_{\alpha 112}^2 + \kappa_{\alpha 122}^2) + 3(2\rho_\alpha^2 + 1) \kappa_{\alpha 112} \kappa_{\alpha 122} + \kappa_{\alpha 111} \kappa_{\alpha 222}. \end{aligned}$$

Note that each u_j is a function of the parameters of the populations and does not depend on the sample size n.

4 Asymptotic non-null distribution of L^*

The power of a statistic is the probability of rejecting the null hypothesis when it is not true. It is the measure of the test procedure's ability to detect deviations from the null case. It is desirable to have a test procedure that provides adequate protection against type 1 error and yet has a high power level from deviations from the null hypothesis. To study the power of a statistic, the non-null distribution is required.

In this section, the asymptotic non-null distribution function of the Kullback's statistic is derived using the asymptotic moments developed in the last section. The distribution function can then be used to make inferences about the power properties of the test procedure based on L^*.

Theorem 4.1. *Let the k sample correlation matrices $R_1, \dots R_k$ be based on samples of sizes $n_1 + 1, \dots n_k + 1$ from populations with finite sixth order cumulants κ^* and population correlation matrices $P_1, \dots P_k$. Define the standardized cumulant $\kappa_\alpha(s_1, s_2, \dots, s_p)$ as*

$$\kappa_\alpha(s_1, s_2, \dots, s_p) = \frac{\kappa^*_\alpha(s_1, s_2, \dots, s_p)}{\sigma_{\alpha 11}\chi_{s_1}\sigma_{\alpha 22}\chi_{s_2}\cdots\sigma_{\alpha pp}\chi_{s_p}},$$

with $\chi_{s_j} = 1$ if $s_j = 0$, $\chi_{s_j} = 1/\sigma^{(\alpha)jj}$ if $s_j \neq 0$ and $\Sigma_\alpha^{-1} = (\sigma^{(\alpha)jj})$.

Let u_1, u_1 and u_3 be as defined in Lemmas 3.2, 3.4 and 3.6, respectively. If $P_\alpha \neq P_\beta$ for at least one pair $\{\alpha, \beta\}$, then the term u_2 is positive and the cumulative distribution function of G is approximated by

$$P\left[\frac{G}{\sqrt{u_2}} \leq x\right] = \Phi(x) - \frac{1}{\sqrt{n}}\phi(x)\left[\frac{u_1}{\sqrt{u_2}} + \frac{(x^2-1)(u_3 - 3u_1u_2)}{6u_2^{3/2}}\right] + O(n^{-1}),$$

where $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Proof. Recall that $u_2 = E(T_1^2)$, where

$$\begin{aligned} E(T_1^2) &= \sum_{\alpha=1}^{k}\sum_{i<j}\sum_{k<\ell}\omega_\alpha \bar{h}^{(1)}_{\alpha ij}\bar{h}^{(1)}_{\alpha k\ell}E(v^{(1)}_{\alpha ij}v^{(1)}_{\alpha k\ell}) \\ &= \sum_{\alpha=1}^{k} vecp(\bar{H}'_\alpha)\Omega_\alpha vecp(\bar{H}_\alpha). \end{aligned}$$

Since Ω_α is a dispersion matrix, it is positive definite. Further, since $P_\alpha \neq P_\beta$ for at least one pair $\{\alpha, \beta\}$, $\bar{H}_\alpha = (\bar{h}_{\alpha ij}) = 2(\bar{P}^{-1} - P_\alpha^{-1}) \neq 0$ for at least one α. Thus $u_2 \neq 0$.

The cumulants of the function G are approximated by using the values u_1, u_2 and u_3, and from the expressions (2.1), (2.2), (2.3) and (2.4). We get

$$\kappa_1 = E(G) = \frac{1}{\sqrt{n}} u_1 + O(n^{-1})$$
$$\kappa_2 = E(G^2) = u_2 - \frac{1}{n} u_1^2 + O(n^{-1}) = u_2 + O(n^{-1})$$
$$\begin{aligned}\kappa_3 &= E(G^3) - 3E(G)E(G^2) + 2[E(G)]^3 \\ &= \frac{1}{\sqrt{n}}(u_3 - 3u_1 u_2) + O(n^{-3/2}).\end{aligned}$$

The estimates of the cumulants of the function G are used to approximate the characteristic function of G. Further,

$$\begin{aligned}E[\exp(\iota t G)] &= \exp\left[(\iota t)\kappa_1 + \frac{1}{2}(\iota t)^2 \kappa_2 + \frac{1}{6}(\iota t)^3 \kappa_3 + \cdots\right] \\ &= \exp\left[\frac{1}{\sqrt{n}}(\iota t) u_1 - \frac{1}{2} t^2 u_2 + \frac{1}{6\sqrt{n}}(\iota t)^3 (u_3 - 3u_1 u_2) + O(n^{-1})\right] \\ &= \exp\Big(-\frac{t^2 u_2}{2}\Big) \exp\left[\frac{1}{\sqrt{n}}\{(\iota t) u_1 + \frac{1}{6}(\iota t)^3 (u_3 - 3u_1 u_2)\} + O(n^{-1})\right] \\ &= \exp\Big(-\frac{t^2 u_2}{2}\Big)\left[1 + \frac{1}{\sqrt{n}}\{(\iota t) u_1 + \frac{1}{6}(\iota t)^3 (u_3 - 3u_1 u_2)\} + O(n^{-1})\right].\end{aligned}$$

Since, u_2 is not zero, the characteristic function for $G/\sqrt{u_2}$ is given by

$$E\left[\exp\Big(\frac{\iota t G}{\sqrt{u_2}}\Big)\right] = \exp\Big(-\frac{t^2}{2}\Big)\left[1 + \frac{1}{\sqrt{n}}\left[(\iota t)\frac{u_1}{\sqrt{u_2}} + \frac{1}{6}(\iota t)^3 \frac{(u_3 - 3u_1 u_2)}{u_2^{3/2}}\right] + O(n^{-1})\right].$$

Hermite polynomials (Boik [9]) may be used to invert this characteristic function. The Hermite polynomial of order j, $H_j(x)$, is defined by

$$(-1)^j \left(\frac{d}{dx}\right)^j \phi(x) = H_j(x)\phi(x).$$

The first four Hermite polynomials are $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, and $H_3(x) = x^3 - 3x$. The Hermite polynomials have the following properties:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\iota t x)(\iota t)^j \exp\left(-\frac{1}{2}t^2\right) dt = H_j(x)\phi(x)$$

and

$$\int_{-\infty}^{x} H_j(t)\phi(t)\,dt = \int_{-\infty}^{x} (-1)^j \left(\frac{d}{dt}\right)^j \phi(t)\,dt = -H_{j-1}(x)\phi(x).$$

Using the first property listed, the characteristic function of $G/\sqrt{u_2}$ is now inverted to obtain an expansion of the probability density function of $G/\sqrt{u_2}$. That is, the density of $G/\sqrt{u_2}$ is derived as

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\iota tx) E\left[\exp\left(\frac{\iota tG}{\sqrt{u_2}}\right)\right] dt \\
&= \phi(x)\left[1 + \frac{1}{\sqrt{n}}\left\{H_1(x)\frac{u_1}{\sqrt{u_2}} + \frac{1}{6}H_3(x)\frac{(u_3 - 3u_1u_2)}{u_2^{3/2}}\right\} + O(n^{-1})\right].
\end{aligned}$$

Using the second property listed for the Hermite polynomials, this is integrated to obtain the approximate cumulative distribution function. □

Although, the complexity of the terms u_1, u_2 and u_3 make the formula in the above theorem rather untenable for power calculation, some conclusions may be drawn from the theorem.

Note that none of u_1, u_2, u_3 depend on the parameter n, thus as the combined sample size increases, the asymptotic distribution of G is normal. From (3.1) and Theorem 3.1, $L^*/\sqrt{n} = \sqrt{n}\,g(P_1, P_2, \ldots, P_k) + G = \sqrt{n}\,g(P_1, P_2, \ldots, P_k) + T_1 + T_2/\sqrt{n} + O_p(n^{-1})$ so that an approximation to the distribution function of the statistic L^* is given by

$$\begin{aligned}
P(L^* < c) &= P\left(\sqrt{n}\,g(P_1, P_2, \ldots, P_k) + T_1 + \frac{1}{\sqrt{n}}T_2 < \frac{c}{\sqrt{n}}\right) \\
&= P\left(\frac{G}{\sqrt{u_2}} < \frac{c - ng(P_1, P_2, \ldots, P_k)}{\sqrt{n}\sqrt{u_2}}\right) \\
&= \Phi\left(\frac{c}{\sqrt{n}\sqrt{u_2}} - \frac{\sqrt{n}g(P_1, P_2, \ldots, P_k)}{\sqrt{u_2}}\right) + O(n^{-1/2}). \qquad (4.1)
\end{aligned}$$

For positive definite matrices A and B, it is well known that $\det(A+B) \geq \det(A)+\det(B)$ so that $g(P_1, P_2, \ldots, P_k) > 0$ in the non-null case. Thus for any fixed value "c", $P(L^* < c) \to 0$ as $\sqrt{n} \to \infty$. This implies that the power of the statistic is related to the square root of the combined sample size and that given enough sample it will be able to detect differences in the underlying population correlation matrices. It is also interesting to note that this is true even if only one of the sample sizes is allowed to go to infinity. Thus the approximation should be valid even if unequal sampling leads to some fairly small samples within populations.

Theorem 4.2. *The test procedure based on L^* is asymptotically unbiased.*

Proof. A test procedure is unbiased if the power function achieves its minimum under the null hypothesis (Anderson [4]). The test procedure based on L^* is to reject the null hypothesis if the observed value is too large. The power function is equal to the probability that the statistic exceeds the given critical value c. Thus the asymptotic power function for L^* is given by

$$\beta(L^*) = 1 - \Phi\left(\frac{c}{\sqrt{n}\sqrt{u_2}} - \frac{\sqrt{n}g(P_1, P_2, \dots, P_k)}{\sqrt{u_2}}\right) + O(n^{-1/2})$$

which is minimized for $g(P_1, P_2, \dots, P_k) = 0$. This condition is satisfied when H is true so that the asymptotic power function achieves its minimum under H. □

Corollary 4.3. *If the samples are taken from bivariate populations which are elliptically contoured with kurtosis given by* $3\kappa_\alpha$*, then the cumulative distribution function of* G *may be approximated by*

$$P\left(\frac{G}{\sqrt{u_2}} < x\right) = \Phi(x) - \frac{1}{\sqrt{n}}\phi(x)\Big[\frac{u_1}{\sqrt{u_2}} + \frac{1}{6}(x^2-1)\frac{(u_3 - 3u_1u_2)}{u_2^{3/2}}\Big] + O(n^{-1}),$$

where

$$\begin{aligned} u_1 &= \sum_{\alpha=1}^{k}\left(\frac{1+\rho_\alpha^2}{(1-\rho_\alpha^2)^2} - \omega_\alpha\frac{1+\bar{\rho}^2}{(1-\bar{\rho}^2)^2}\right)[(1-\rho_\alpha^2)^2 + (1+2\rho_\alpha^2)\kappa_\alpha] \\ &\quad + \sum_{\alpha=1}^{k}\left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2}\right)(\rho_\alpha^3 - \rho_\alpha + 5\rho_\alpha\kappa_\alpha), \\ u_2 &= 4\sum_{\alpha=1}^{k}\omega_\alpha\left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2}\right)^2[(1-\rho_\alpha^2)^2 + (1+2\rho_\alpha^2)\kappa_\alpha] \end{aligned}$$

and

$$u_3 = 144\sum_{\alpha=1}^{k}\omega_\alpha\left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2}\right)^3(\rho_\alpha^2 - 1)\rho_\alpha\kappa_\alpha.$$

Proof. For elliptically contoured distributions the fourth order cumulants are such that $\kappa_{iiii} = 3\kappa_{iijj} = 3\kappa$ for $i \neq j$ and all other cumulants are zero (Waternaux [21], Gupta, Varga and Bodnar [11]). Deleting the appropriate terms in corollaries 3.3, 3.5 and 3.7 leaves the above expressions for u_1, u_2 and u_3. The result then follows from Theorem 4.1. □

Corollary 4.4. *If the samples are taken from bivariate normal populations, then the cumulative distribution function of* G *may be approximated by*

$$P\left(\frac{G}{\sqrt{u_2}} < x\right) = \Phi(x) - \frac{1}{2\sqrt{n}}\phi(x)\frac{u_1}{\sqrt{u_2}}(3 - x^2) + O(n^{-1}),$$

where

$$u_1 = \sum_{\alpha=1}^{k} \left(\frac{1+\rho_\alpha^2}{(1-\rho_\alpha^2)^2} - \omega_\alpha \frac{1+\bar{\rho}^2}{(1-\bar{\rho}^2)^2} \right) (1-\rho_\alpha^2)^2 + \sum_{\alpha=1}^{k} \left(\frac{\rho_\alpha}{1-\rho_\alpha^2} + \frac{\bar{\rho}}{1-\bar{\rho}^2} \right) (\rho_\alpha^3 - \rho_\alpha)$$

and

$$u_2 = 4 \sum_{\alpha=1}^{k} \omega_\alpha \left(\frac{\rho_\alpha}{1-\rho_\alpha^2} - \frac{\bar{\rho}}{1-\bar{\rho}^2} \right)^2 (1-\rho_\alpha^2)^2.$$

Proof. For normal populations, all cumulants of order three and higher are zero (Anderson [4]). Deleting the appropriate terms in Corollary 4.3 leaves the above expressions for u_1 and u_2 while u_3 vanishes entirely. The result then follows upon simplification. □

Corollaries 4.3 and 4.4 may be used to get a better understanding of the relationship between the power of the test and the relative differences between the population parameters. From expression (4.1), we have

$$P(L^* > c) = P\left(\frac{G}{\sqrt{u_2}} > \frac{c}{\sqrt{n}\sqrt{u_2}} - \frac{\sqrt{n}g(P_1, P_2, \ldots, P_k)}{\sqrt{u_2}} \right).$$

For the case $k = 2$, a level α test would reject the null hypothesis if the observed statistic exceeds C where $P(L^* > C) < \alpha$. From Lemma 2.1, this is equivalent to a value such that

$$P\left(L^* > \chi_\alpha^2(1)[(1-\rho^2)^2 + (1+2\rho^2)\kappa] \frac{1+\rho^2}{(1-\rho^2)^2} \right),$$

where ρ is the common correlation coefficient and $\chi_\alpha^2(1)$ is the $1-\alpha$ percentile of the chi-square distribution with one degree of freedom.

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