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Bivariate Topp-Leone family of distributions

Daya K. Nagar, Edwin Zarrazola, Santiago Echeverri-Valencia

Instituto de Matemáticas
Universidad de Antioquia
Calle 67, No. 53–108
Medellín, Colombia

email: dayaknagar@yahoo.com, edwin.zarrazola@udea.edu.co, santiago.echeverriv@udea.edu.co

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Abstract

The Topp-Leone family of univariate distributions has drawn considerable attention in recent years. In this article, we define a bivariate generalization of the Topp-Leone distribution. Moreover, we derive several results such as marginal and conditional distributions, joint moments, correlation coefficient, and Fisher information matrix. Furthermore, we derive the exact distributions of $X+Y$, $X/(X+Y)$ and XY when X and Y follow bivariate Topp-Leone distribution.

1 Introduction

Consider the probability distribution defined by the probability density function (pdf)

$$f(x;\nu,\sigma) = \frac{2\nu}{\sigma}\left(\frac{x}{\sigma}\right)^{\nu-1}\left(1-\frac{x}{\sigma}\right)\left(2-\frac{x}{\sigma}\right)^{\nu-1}, \quad 0<x<\sigma<\infty. \qquad (1.1)$$

The above family of distributions was introduced by Topp and Leone [28] over 50 years ago and is called *Topp-Leone family of distributions*.

For $0<\nu<1$, the distribution defined by the density (1.1) is referred to as the J-shaped distribution by Topp and Leone [28] because $f(x;\nu,\sigma)>0$,

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$f'(x;\nu,\sigma) < 0$ and $f''(x;\nu,\sigma) > 0$ for all $0 < x < \sigma$, where $f'(x;\nu,\sigma)$ is the first derivative of $f(x;\nu,\sigma)$ and $f''(x;\nu,\sigma)$ is the second derivative of $f(x;\nu,\sigma)$. For $\nu > 1$, (1.1) is still a density and attains different shapes depending on values of parameters (see Kotz and van Dorp [13]). Therefore, we will not restrict ν to $(0,1)$ and assume ν to be positive taking values in $(0,\infty)$. Further, we will write $X \sim \text{TL}(\nu;\sigma)$ if the density of X is given by (1.1).

In recent years, the Top-Leone family of probability distributions has received renewed interest and several papers have appeared dealing with various aspects of the univariate T-L distribution. Nadarajah and Kotz [19] provided explicit algebraic expressions for hazard rate function, raw and central moments and characteristic function of the Topp-Leone distribution. Systematic treatment of Topp-Leone distribution was given by Kotz and van Dorp [13, Chapter 2] who studied a number of results. Kotz and van Dorp [13, Chapter 7] have also studied the reflected generalized Topp-Leone distribution. Kotz and Seier [14] have examined the kurtosis of the T-L family of distributions for three intervals of values of the parameter ν: $0 < \nu < 1$, $1 < \nu < 2$, and $\nu \geq 2$. For further insight into the T-L distribution, see [1], [2], [4], [7], [8], [18], [30], and [31, 32].

In the last two decades, although several variations of the T-L distribution have been proposed and studied, not much work has been done in the area of bivariate T-L distributions.

In this article, we present a bivariate generalization of the Topp-Leone distribution and study some of its properties including marginal and conditional distributions, joint moments and the coefficient of correlation. Further, for different values of parameters, we give graphically a variety of forms of the bivariate Topp-Leone density. In addition to these results, we also derive densities of several basic transformations such as $X + Y$, $X/(X + Y)$, X/Y, and XY of two variables X and Y jointly distributed as bivariate Topp-Leone.

The bivariate Topp-Leone distribution studied in this article is a new member of the multivariate Liouville family of distributions. Like Dirichlet, Liouville and other bivariate beta distributions, this distribution can also be used in Bayesian analysis, modeling of bivariate data such as proportions of substances in a mixture, proportions of brands of some consumer product that are bought by customers, proportions of the electorate voting for the candidate in a two-candidate election, drought duration and drought intensity in climate science, the dependence between two soil strength parameters, and reliability theory. These distributions can also be used in geology, biology, chemistry, forensic science, and statistical genetics.

For a review of known bivariate and matrix variate distributions the reader may consult excellent texts by Mardia [17], Kotz, Balakrishnan and Johnson [12], Gupta and Nagar [10], and Balakrishnan and Lai [3]. For some recent work on bivariate distributions the reader is referred to Nadarajah, Shih and Nagar [20], Nagar, Morán-Vásquez and Roldán-Correa [21], Nagar, Nadarajah and Okorie [22], Nagar, Zarrazola and Roldán-Correa [24, 25], Orozco-Castañeda, Nagar and Gupta [26], Ghosh [9], Rafiei, Iranmanesh and Nagar [29], and references therein.

In Section 2, we have defined a bivariate generalization of the Topp-Leone distribution. This section also suggests bivariate Topp-Leone distribution as prior for Bayesian analyses involving multinomial proportions. Several properties of this distribution including marginal distributions, joint and marginal moments and correlation coefficient are derived in Section 3. In sections 4 and 5, exact distributions of $X+Y$, $X/(X+Y)$ and XY and the Fisher information matrix are derived when X and Y follow the bivariate distribution proposed in this article. In Section 6, a resumé of results derived in this article is given and a multivariate generalization of the Topp-Leone distribution is also proposed. Finally, in the the Appendix we list a number definitions and known results.

2 The Density Function

A bivariate generalization of the Topp-Leone distribution can be defined as follows:

Definition 2.1. *The random variables X and Y are said to have a bivariate Topp-Leone distribution with parameters $\nu_i > 0$, $i = 1, 2$ and $\sigma > 0$, denoted by $(X, Y) \sim \mathrm{BTL}(\nu_1, \nu_2; \sigma)$, if their joint pdf is given by*

$$f(x, y; \nu_1, \nu_2, \sigma) = C(\nu_1, \nu_2, \sigma) x^{\nu_1 - 1} y^{\nu_2 - 1} \left(1 - \frac{x+y}{\sigma}\right) \left(2 - \frac{x+y}{\sigma}\right)^{\nu_1+\nu_2-1}, \tag{2.2}$$

where $x > 0$, $y > 0$ and $x + y < \sigma$.

Integrating the joint density of X and Y over its support $\{(x, y) : x > 0, y > 0, x + y < \sigma\}$ and using (A.2), the normalizing constant $C(\nu_1, \nu_2, \sigma)$ is evaluated as

$$[C(\nu_1, \nu_2, \sigma)]^{-1} = \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1 + \nu_2)} \int_0^\sigma z^{\nu_1+\nu_2-1} \left(1 - \frac{z}{\sigma}\right) \left(2 - \frac{z}{\sigma}\right)^{\nu_1+\nu_2-1} \mathrm{d}z$$

$$= \frac{\sigma^{\nu_1+\nu_2}\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_2)} \int_0^1 t\left(1-t^2\right)^{\nu_1+\nu_2-1} \mathrm{d}t,$$

where we have used the substitution $t = 1 - z/\sigma$. Now, evaluating the above integral by using the definition of beta function, we obtain

$$C(\nu_1, \nu_2, \sigma) = \frac{2\Gamma(\nu_1+\nu_2+1)}{\sigma^{\nu_1+\nu_2}\Gamma(\nu_1)\Gamma(\nu_2)}.$$

For $\sigma = 1$, the density in (2.2) reduces to

$$C(\nu_1, \nu_2, 1)\, x^{\nu_1-1} y^{\nu_2-1} (1-x-y)\,(2-x-y)^{\nu_1+\nu_2-1}, \tag{2.3}$$

where $x > 0$, $y > 0$ and $x + y < 1$, and in this case we write $(X, Y) \sim \mathrm{BTL}(\nu_1, \nu_2)$.

In Bayesian probability theory, if the posterior distribution is in the same family as the prior probability distribution; the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior. In case of multinomial distribution, the usual conjugate prior is the Dirichlet distribution. If

$$P(s_1, s_2, f|x_1, x_2) = \binom{s_1+s_2+f}{s_1, s_2, f} x_1^{s_1} x_2^{s_2} (1-x_1-x_2)^f$$

and

$$p(x_1, x_2) = \frac{2\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1)\Gamma(\nu_2)}\, x_1^{\nu_1-1} x_2^{\nu_2-1} (1-x_1-x_2)(2-x_1-x_2)^{\nu_1+\nu_2-1},$$

where $x_1 > 0$, $x_2 > 0$, and $x_1 + x_2 < 1$, then

$$\begin{aligned} P(s_1, s_2, f) &= \binom{s_1+s_2+f}{s_1, s_2, f} \frac{2\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 \int_0^{1-x_2} x_1^{\nu_1+s_1-1} x_2^{\nu_2+s_2-1} \\ &\quad \times (1-x_1-x_2)^{f+1} (2-x_1-x_2)^{\nu_1+\nu_2-1}\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \binom{s_1+s_2+f}{s_1, s_2, f} \frac{2\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\nu_1+s_1)\Gamma(\nu_2+s_2)}{\Gamma(\nu_1+\nu_2+s_1+s_2)} \\ &\quad \times \int_0^1 x^{\nu_1+\nu_2+s_1+s_2-1} (1-x)^{f+1} (2-x)^{\nu_1+\nu_2-1}\, \mathrm{d}x, \end{aligned}$$

where we have used the Liouville-Dirichlet integral (A.2). Now, substituting $t = 1 - x$, expanding $(1-t)^{s_1+s_2}$ and integrating the resulting expression by using the definition of beta function, we obtain

$$P(s_1, s_2, f) = \binom{s_1+s_2+f}{s_1, s_2, f} \frac{2\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\nu_1+s_1)\Gamma(\nu_2+s_2)}{\Gamma(\nu_1+\nu_2+s_1+s_2)}$$

$$\times \frac{1}{2}\sum_{r=0}^{s_1+s_2}(-1)^r\binom{s_1+s_2}{r}\frac{\Gamma[1+(f+r)/2]\Gamma(\nu_1+\nu_2)}{\Gamma[\nu_1+\nu_2+1+(f+r)/2]}.$$

Finally, the posterior distribution is derived as

$$p(x_1,x_2|s_1,s_2,f) = \frac{2\Gamma(\nu_1+\nu_2+s_1+s_2)}{\Gamma(\nu_1+\nu_2)\Gamma(\nu_1+s_1)\Gamma(\nu_2+s_2)} \times \frac{x_1^{\nu_1+s_1-1}x_2^{\nu_2+s_2-1}(1-x_1-x_2)^{f+1}(2-x_1-x_2)^{\nu_1+\nu_2-1}}{\sum_{r=0}^{s_1+s_2}(-1)^r\binom{s_1+s_2}{r}\frac{\Gamma[1+(f+r)/2]}{\Gamma[\nu_1+\nu_2+1+(f+r)/2]}}.$$

Observe that the distribution defined by the above density is a generalization of the bivariate T-L distribution.

3 Properties

A distribution is said to be negatively likelihood ratio dependent (NLRD) if the density $f(x,y)$ satisfies

$$f(x_1,y_1)f(x_2,y_2) \le f(x_1,y_2)f(x_2,y_1) \tag{3.4}$$

for all $x_1 > x_2$ and $y_1 > y_2$ (Lehmann [15], Tong [27]). In the present case (3.4) is equivalent to

$$\left(1-\frac{x_1+y_1}{\sigma}\right)\left(1-\frac{x_2+y_2}{\sigma}\right)\left[\left(1-\frac{x_1+y_1}{\sigma}\right)\left(1-\frac{x_2+y_2}{\sigma}\right)\right]^{\nu_1+\nu_2-1}$$
$$\le \left(1-\frac{x_1+y_2}{\sigma}\right)\left(1-\frac{x_2+y_1}{\sigma}\right)\left[\left(1-\frac{x_1+y_2}{\sigma}\right)\left(1-\frac{x_2+y_1}{\sigma}\right)\right]^{\nu_1+\nu_2-1}$$

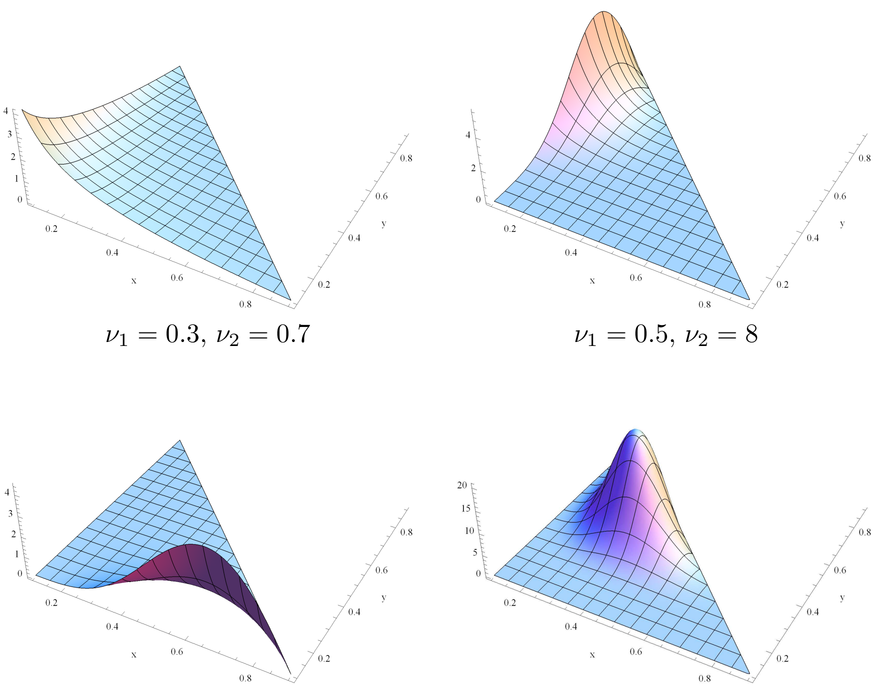
which clearly holds if $\nu_1+\nu_2 \ge 1$. Thus, the bivariate distribution defined by the density (2.2), for $\nu_1+\nu_2 \ge 1$, is NLRD.

In continuation, we present a few graphs (Figure 1, Figure 2) of the density function defined by the expression (2.2) for different values of parameters. Here one can appreciate the wide range of forms that result from the bivariate Topp-Leone distribution.

Next, we derive marginal densities of X and Y.

Theorem 3.1. *Let $(X,Y) \sim \text{BTL}(\nu_1,\nu_2;\sigma)$. Then, the marginal density of X is*

$$\frac{2\Gamma(\nu_1+\nu_2+1)}{\sigma^{\nu_1}\Gamma(\nu_1)\Gamma(\nu_2+2)}x^{\nu_1-1}\left(1-\frac{x}{\sigma}\right)^{\nu_2+1}F\left(2,-\nu_1-\nu_2+1;\nu_2+2;-\left(1-\frac{x}{\sigma}\right)\right),$$



$\nu_1 = 0.3$, $\nu_2 = 0.7$ $\nu_1 = 0.5$, $\nu_2 = 8$

$\nu_1 = 7$, $\nu_2 = 0.4$ $\nu_1 = 10$, $\nu_2 = 15$

Figure 1: Graphs of the bivariate Topp-Leone density for $\sigma = 1$.

where $0 < x < \sigma$*. Further, the marginal density of* Y *is*

$$\frac{2\Gamma(\nu_1+\nu_2+1)}{\sigma^{\nu_2}\Gamma(\nu_1+2)\Gamma(\nu_2)} y^{\nu_2-1}\left(1-\frac{y}{\sigma}\right)^{\nu_1+1} F\left(2, -\nu_1-\nu_2+1; \nu_1+2; -\left(1-\frac{y}{\sigma}\right)\right),$$

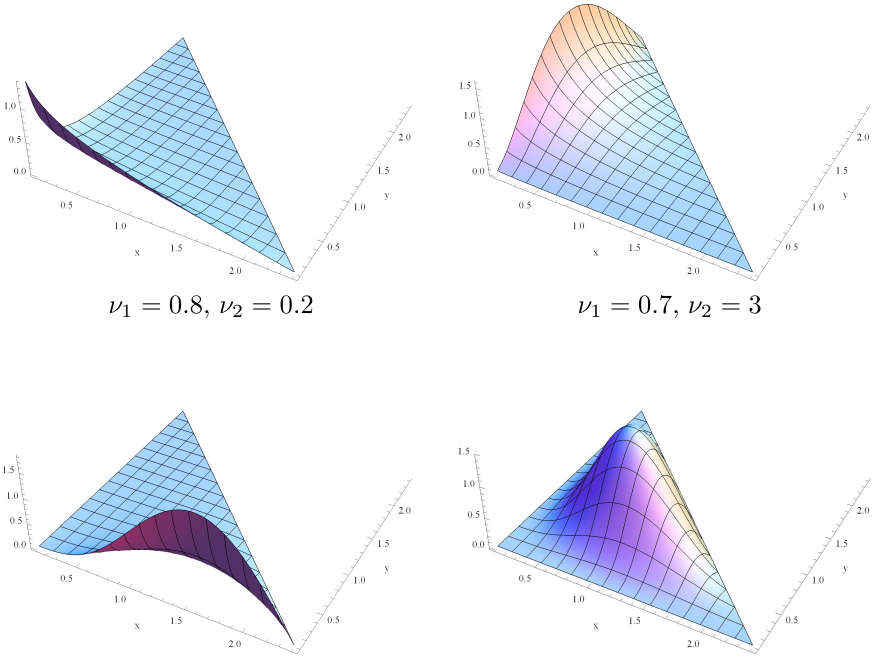
where $0 < y < \sigma$*.*

Proof. Integrating the density (2.2) with respect to y, the marginal density of X is derived as

$$C(\nu_1,\nu_2,\sigma)\; x^{\nu_1-1}\int_0^{\sigma-x} y^{\nu_2-1}\left(1-\frac{x+y}{\sigma}\right)\left(2-\frac{x+y}{\sigma}\right)^{\nu_1+\nu_2-1} \mathrm{d}y$$

$$= C(\nu_1,\nu_2,\sigma) x^{\nu_1-1}\left(1-\frac{x}{\sigma}\right)^{\nu_2+1}\int_0^1 z^{\nu_2-1}(1-z)\left[1+\left(1-\frac{x}{\sigma}\right)(1-z)\right]^{\nu_1+\nu_2-1} \mathrm{d}z,$$

where we have substituted $z = y/(\sigma - x)$. Finally, using the integral representation of the Gauss hypergeometric function given in (A.1), the desired result is obtained. Similarly, the marginal density of Y can be derived. □



$\nu_1 = 0.8$, $\nu_2 = 0.2$ $\nu_1 = 0.7$, $\nu_2 = 3$

$\nu_1 = 5$, $\nu_2 = 0.4$ $\nu_1 = 5$, $\nu_2 = 5$

Figure 2: Graphs of the bivariate Topp-Leone density for $\sigma = 2.5$.

Using the marginal density of Y, we obtain the conditional density function of the random variable X given $Y = y$, $0 < y < \sigma$, as

$$\frac{\nu_1(\nu_1+1)x^{\nu_1-1}[1-(x+y)/\sigma][2-(x+y)/\sigma]^{\nu_1+\nu_2-1}}{\sigma^{\nu_1}(1-y/\sigma)^{\nu_1+1}F\left(2,-\nu_1-\nu_2+1,\nu_1+2,-(1-y/\sigma)\right)}, \quad 0 < x < \sigma - y.$$

Additionally, for non-negative integers r and s, the (r,s)-the joint moment of X and Y, $(X,Y) \sim \mathrm{BTL}(\nu_1,\nu_2;\sigma)$, can be expressed as

$$\begin{aligned} \mathrm{E}(X^rY^s) &= \frac{\Gamma(\nu_1+\nu_2+1)\Gamma(\nu_1+r)\Gamma(\nu_2+s)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_1+\nu_2+r+s)}\sigma^{r+s} \\ &\quad \times \sum_{k=0}^{r+s}(-1)^k\binom{r+s}{k}\frac{\Gamma(1+k/2)\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1+\nu_2+1+k/2)}. \end{aligned} \tag{3.5}$$

Substituting appropriately for r and s in (3.5) and using proper definitions, we obtain

$$\mathrm{E}(X) = \frac{\nu_1\sigma}{\nu_1+\nu_2}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{2\Gamma(\nu_1+\nu_2+3/2)}\right],$$

$$\mathrm{E}(Y) = \frac{\nu_2\sigma}{\nu_1+\nu_2}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{2\Gamma(\nu_1+\nu_2+3/2)}\right],$$

$$\mathrm{E}(X^2) = \frac{\nu_1(\nu_1+1)\sigma^2}{(\nu_1+\nu_2)(\nu_1+\nu_2+1)}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1+\nu_2+3/2)} + \frac{1}{\nu_1+\nu_2+1}\right],$$

$$\mathrm{E}(Y^2) = \frac{\nu_2(\nu_2+1)\sigma^2}{(\nu_1+\nu_2)(\nu_1+\nu_2+1)}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1+\nu_2+3/2)} + \frac{1}{\nu_1+\nu_2+1}\right],$$

$$\mathrm{E}(XY) = \frac{\nu_1\nu_2\sigma^2}{(\nu_1+\nu_2)(\nu_1+\nu_2+1)}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1+\nu_2+3/2)} + \frac{1}{\nu_1+\nu_2+1}\right],$$

$$\begin{aligned}\mathrm{Var}(X) = \frac{\nu_1\sigma^2}{\nu_1+\nu_2}\Bigg\{&\frac{\nu_1+1}{\nu_1+\nu_2+1}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1+\nu_2+3/2)} + \frac{1}{\nu_1+\nu_2+1}\right] \\ &- \frac{\nu_1}{\nu_1+\nu_2}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{2\Gamma(\nu_1+\nu_2+3/2)}\right]^2\Bigg\},\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Y) = \frac{\nu_2\sigma^2}{\nu_1+\nu_2}\Bigg\{&\frac{\nu_2+1}{\nu_1+\nu_2+1}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1+\nu_2+3/2)} + \frac{1}{\nu_1+\nu_2+1}\right] \\ &- \frac{\nu_2}{\nu_1+\nu_2}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{2\Gamma(\nu_1+\nu_2+3/2)}\right]^2\Bigg\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X,Y) = \frac{\nu_1\nu_2\sigma^2}{\nu_1+\nu_2}\Bigg\{&\frac{1}{\nu_1+\nu_2+1}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{\Gamma(\nu_1+\nu_2+3/2)} + \frac{1}{\nu_1+\nu_2+1}\right] \\ &- \frac{1}{\nu_1+\nu_2}\left[1 - \frac{\sqrt{\pi}\,\Gamma(\nu_1+\nu_2+1)}{2\Gamma(\nu_1+\nu_2+3/2)}\right]^2\Bigg\}.\end{aligned}$$

Using expressions for variances and covariance, the correlation coefficient between X and Y can be calculated by using the formula

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{[\mathrm{Var(X)Var(Y)}]^{1/2}}.$$

Table 1 contains numerical values of the correlation between X and Y for different values of parameters. Since the coefficient of correlation is independent of change of scale, we can take $\sigma = 1$. As expected, all the values in the table are negative because of the condition $x + y < 1$. Further, by selecting properly values of parameters, it is possible to find values of the correlation close to 0 or -1. For example for ν_i small and ν_j large, $i \neq j$, the correlation is close to zero and for large values of both the parameters the correlation is close to -1. Graphs in Figure 3 illustrate the behavior of the correlation as a function of one of the parameters.

Table 1: Correlations between X and Y for different values of ν_1 and ν_2

ν_1	ν_2						
	0.5	1	2	3	5	10	20
0.10	–0.106	–0.124	–0.131	–0.13	–0.123	–0.106	–0.0865
0.25	–0.157	–0.184	–0.198	–0.197	–0.188	–0.165	–0.135
0.50	–0.2	–0.238	–0.261	–0.263	–0.254	–0.226	–0.188
1	–0.238	–0.290	–0.326	–0.334	–0.33	–0.301	–0.256
2	–0.261	–0.326	–0.379	–0.398	–0.405	–0.385	–0.339
3	–0.263	–0.334	–0.398	–0.425	–0.442	–0.433	–0.391
5	–0.254	–0.33	–0.405	–0.442	–0.474	–0.484	–0.455
10	–0.226	–0.301	–0.385	–0.433	–0.484	–0.525	–0.525
15	–0.204	–0.276	–0.36	–0.412	–0.471	–0.53	–0.551
20	–0.188	–0.256	–0.339	–0.391	–0.455	–0.525	–0.561
25	–0.175	–0.24	–0.32	–0.372	–0.439	–0.517	–0.564

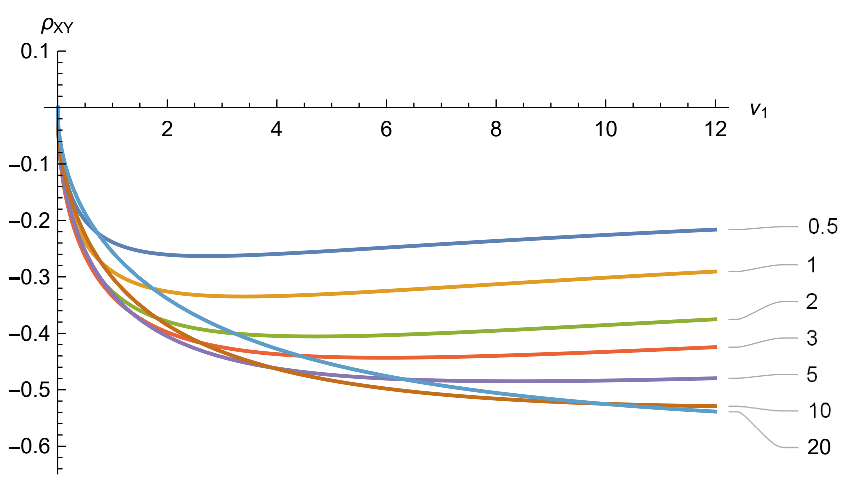


Figure 3: Graphs of ρ_{XY} for $\nu_2 = 0.5, 1, 2, 3, 5, 10, 20$

4 Some Transformations

In this section, we obtain expressions for densities of sum, quotient and product of two random variables whose joint density is given by (2.2).

Theorem 4.1. *Let* $(X,Y) \sim \mathrm{BTL}(\nu_1,\nu_2;\sigma)$ *and define* $S = X+Y$, *and* $U = X/(X+Y)$. *Then,* S *and* U *are independent,* $U \sim \mathrm{B1}(\nu_1,\nu_2)$ *and* $S \sim \mathrm{TL}(\nu_1+\nu_2;\sigma)$.

Proof. Transforming $S = X+Y$ and $U = X/(X+Y)$ with the Jacobian $J(x,y \to s,u) = s$ in (2.2), the joint density of S and U is given by

$$\frac{2\Gamma(\nu_1+\nu_2+1)}{\sigma^{\nu_1+\nu_2}\Gamma(\nu_1)\Gamma(\nu_2)}\, u^{\nu_1-1}(1-u)^{\nu_2-1}s^{\nu_1+\nu_2-1}\left(1-\frac{s}{\sigma}\right)\left(2-\frac{s}{\sigma}\right)^{\nu_1+\nu_2-1},$$

where $0<s<1$ and $0<u<1$. From the above factorization, it is clear that U and S are independent, $U \sim \mathrm{B1}(\nu_1,\nu_2)$ and $S \sim \mathrm{TL}(\nu_1+\nu_2;\sigma)$. □

Corollary 4.2. *If* $(X,Y) \sim \mathrm{BTL}(\nu_1,\nu_2;\sigma)$, *then* $X/Y \sim \mathrm{B2}(\nu_1,\nu_2)$.

Corollary 4.3. *If* $(X,Y) \sim \mathrm{BTL}(\nu_1,\nu_2;\sigma)$, *then*

$$\mathrm{E}\left[\left(\frac{X+Y}{\sigma}\right)\left(2-\frac{X+Y}{\sigma}\right)\right]^k = \frac{\nu_1+\nu_2}{\nu_1+\nu_2+k}.$$

Theorem 4.4. *If* $(X,Y) \sim \mathrm{BTL}(\nu_1,\nu_2;\sigma)$ *and* $X+Y=S$, *then*

$$\mathrm{E}\left[\frac{(S/\sigma)^k}{(2-S/\sigma)^\ell}\right] = (\nu_1+\nu_2)\sum_{j=0}^{k+\ell}\binom{k+\ell}{j}(-1)^j\frac{\Gamma(1+j/2)\Gamma(\nu_1+\nu_2-\ell)}{\Gamma(\nu_1+\nu_2-\ell+1+j/2)},$$

where $\nu_1+\nu_2-\ell>0$.

Proof. From Theorem 4.1, we know that $X+Y=S \sim \mathrm{TL}(\nu;\sigma)$, where $\nu = \nu_1+\nu_2$. Now, by definition

$$\mathrm{E}\left[\frac{(S/\sigma)^k}{(2-S/\sigma)^\ell}\right] = \frac{2\nu}{\sigma}\int_0^\sigma\left(\frac{s}{\sigma}\right)^{\nu+k-1}\left(1-\frac{s}{\sigma}\right)\left(2-\frac{s}{\sigma}\right)^{\nu-\ell-1}\mathrm{d}s.$$

Substitution $t = 1-s/\sigma$ in the above expression, one gets

$$\mathrm{E}\left[\frac{(S/\sigma)^k}{(2-S/\sigma)^\ell}\right] = 2\nu\int_0^1 (1-t)^{k+\ell}\,t\left(1-t^2\right)^{\nu-\ell-1}\mathrm{d}t.$$

Now, expanding $(1-t)^{k+\ell}$ using binomial theorem and integrating t, we get

$$\mathrm{E}\left[\frac{(S/\sigma)^k}{(2-S/\sigma)^\ell}\right] = \nu\sum_{j=0}^{k+\ell}\binom{k+\ell}{j}(-1)^j\frac{\Gamma(1+j/2)\Gamma(\nu-\ell)}{\Gamma(\nu-\ell+1+j/2)},$$

where $\nu-\ell>0$. □

Corollary 4.5. *If* $(X,Y)\sim \mathrm{BTL}(\nu_1,\nu_2;\sigma)$ *and* $X+Y=S$, *then*

$$\mathrm{E}\left[\frac{S/\sigma}{2-S/\sigma}\right]^k = \nu\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j\frac{\Gamma(1+j/2)\Gamma(\nu-k)}{\Gamma(\nu-k+1+j/2)},$$

where $\nu-k>0$ *and*

$$\mathrm{E}\left[\frac{S}{\sigma}\right]^k = \nu\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{\Gamma(1+j/2)\Gamma(\nu)}{\Gamma(\nu+1+j/2)}.$$

Substituting $(k,\ell)=(1,1)$ and $(k,\ell)=(2,2)$ in the above corollary and simplifying resulting expressions,

$$\mathrm{E}\left[\frac{S/\sigma}{2-S/\sigma}\right] = \frac{\nu+1}{\nu-1}-\frac{\sqrt{\pi}\nu\Gamma(\nu-1)}{\Gamma\left(\nu+1/2\right)},\quad \nu>1,$$

$$\mathrm{E}\left[\frac{S/\sigma}{2-S/\sigma}\right]^2 = \frac{\nu^2+5\nu+2}{(\nu-1)(\nu-2)}-\frac{2\sqrt{\pi}(\nu+1)\nu\Gamma(\nu-2)}{\Gamma\left(\nu+1/2\right)},\quad \nu>2.$$

Similarly, one can easily derive

$$\mathrm{E}\left[\frac{S/\sigma}{1-S/\sigma}\right] = \frac{\sqrt{\pi}\,\Gamma(\nu+1)}{\Gamma(\nu+1/2)}-1$$

and

$$\mathrm{E}\left[\frac{S/\sigma}{1-S/\sigma}\right]^2 = \sum_{k=0}^{\infty}\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^j\frac{\Gamma(1+j/2)\Gamma(\nu+1)}{\Gamma(\nu+1+j/2)}.$$

Nadarajah and Kotz [19] have also given an expression for integer order moments of the Topp-Leone variable in terms of beta functions and factorials of negative number. However, expressions given here are much compact and simpler to use.

The distribution of XY has been studied by several authors especially when X and Y are independent random variables and come from the same family (for example see Nagar and Zarrazola [23]). However, there is relatively little work of this kind when X and Y are correlated random variables. For a bivariate random vector (X,Y), the distribution of the product XY is of interest in problems in biological and physical sciences, econometrics, and classification. The following theorem gives a partial result on the distribution of the product of two random variables distributed jointly as bivariate Topp-Leone.

Theorem 4.6. *Let $(X,Y) \sim \text{BTL}(\nu_1,\nu_2)$. If $\nu_1+\nu_2 = i$ is an integer, then the pdf of the random variable $Z = XY$, is given by*

$$\begin{aligned} &C(\nu_1,\nu_2,1)\frac{(b-a)^3[(b-p)(q-b)]^{\nu_1+\nu_2-1}}{b^{2\nu_2+1}} z^{\nu_2-1} \\ &\times \sum_{r=0}^{i-1}\sum_{s=0}^{i-1}\binom{i-1}{r}\binom{i-1}{s}\frac{(b-a)^{r+s}}{(b-p)^r(b-q)^s} \\ &\times \frac{\Gamma(r+s+2)}{\Gamma(r+s+4)} F\left(r+s+2, 2\nu_2+1; r+s+4; \frac{b-a}{b}\right), \quad z < 1/4, \end{aligned}$$

where

$$a = \frac{1-\sqrt{1-4z}}{2},\ b = \frac{1+\sqrt{1-4z}}{2},\ p = -1-\sqrt{1-z},\ q = -1+\sqrt{1-z}\,.$$

Proof. We consider the transformation $X = X$ and $Z = XY$. Given that $x + y < 1$ and $y = z/x$, we have $x^2 - x + z < 0$. In terms of a and b, these conditions can be expressed as $a < x < b$ and $z < 1/4$. Further, the Jacobian of this transformation is $J(x, y \to x, z) = 1/x$. Thus, substituting appropriately in (2.3), the joint pdf of X and Z is given by

$$C(\nu_1,\nu_2,1) z^{\nu_2-1} x^{-2\nu_2-1}(-x^2+x-z)(-x^2+2x-z),\ a<x<b,\ z<1/4,$$

Now, integrating with respect to x, we get the marginal density of Z as

$$\begin{aligned} &C(\nu_1,\nu_2,1)z^{\nu_2-1}\int_a^b \frac{(-x^2+x-z)(-x^2+2x-z)^{\nu_1+\nu_2-1}}{x^{2\nu_2+1}}\,\mathrm{d}x \\ &= C(\nu_1,\nu_2,1)z^{\nu_2-1}\int_a^b \frac{[-(x-a)(x-b)][-(x-p)(x-q)]^{\nu_1+\nu_2-1}}{x^{2\nu_2+1}}\,\mathrm{d}x,\ z<1/4. \end{aligned}$$

Since $a < x < b$, we have $0 < (x-b)/(a-b) < 1$. Further, if $u = (x-b)/(a-b)$, then $x = (a-b)u + b$ and $\mathrm{d}x = (a-b)\mathrm{d}u$. Furthermore, $(x-b) = (a-b)u$ and $x - a = (b-a)(1-u)$, and the above density is expressed as

$$C(\nu_1,\nu_2,1)z^{\nu_2-1}\frac{(b-a)^3[(b-p)(q-b)]^{\nu_1+\nu_2-1}}{b^{2\nu_2+1}}$$

$$\times \int_0^1 \frac{u(1-u)\{[1-(b-a)u/(b-p)][1-(b-a)u/(b-q)]\}^{\nu_1+\nu_2-1}}{[1-(b-a)u/b]^{2\nu_2+1}} \mathrm{d}u.$$

For $\nu_1 + \nu_2 = i$, where i is a positive integer, writing

$$\left[1 - \frac{(b-a)u}{b-p}\right]^{i-1} \left[1 - \frac{(b-a)u}{b-q}\right]^{i-1}$$
$$= \sum_{r=0}^{i-1} \sum_{s=0}^{i-1} \binom{i-1}{r} \binom{i-1}{s} \frac{(b-a)^{r+s} u^{r+s}}{(b-p)^r (b-q)^s}$$

and using the integral representation of the Gauss hypergeometric function given in (A.1), the above integral is evaluated as

$$\sum_{r=0}^{i-1} \sum_{s=0}^{i-1} \binom{i-1}{r} \binom{i-1}{s} \frac{(b-a)^{r+s}}{(b-p)^r (b-q)^s}$$
$$\times \frac{\Gamma(r+s+2)}{\Gamma(r+s+4)} F\left(r+s+2, 2\nu_2+1; r+s+4; \frac{b-a}{b}\right).$$

Now, substituting appropriately, we get the desired result. □

5 Fisher Information Matrix

In this section, we calculate the Fisher information matrix for the bivariate distribution defined by the density (2.2). The information matrix plays a significant role in statistical inference in connection with estimation, sufficiency and properties of variances of estimators. For a given observation vector (x, y), the Fisher information matrix for the bivariate distribution defined by the density (2.2) is defined as

$$-\begin{bmatrix} E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_1^2}\right) & E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_1 \partial \nu_2}\right) & E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_1 \partial \sigma}\right) \\ E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_2 \partial \nu_1}\right) & E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_2^2}\right) & E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_2 \partial \sigma}\right) \\ E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma \partial \nu_1}\right) & E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma \partial \nu_2}\right) & E\left(\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma^2}\right) \end{bmatrix},$$

where $L(\nu_1, \nu_2, \sigma) = \ln f(x, y; \nu_1, \nu_2, \sigma)$. From (2.2), the natural logarithm of $L(\nu_1, \nu_2, \sigma)$ is obtained as

$$\ln L(\nu_1,\nu_2,\sigma) = \ln 2 + \ln \Gamma(\nu_1+\nu_2+1) - \ln \Gamma(\nu_1) - \ln \Gamma(\nu_2) - (\nu_1+\nu_2) \ln \sigma$$

$$+ (\nu_1 - 1)\ln x + (\nu_2 - 1)\ln y + \ln\left(1 - \frac{x+y}{\sigma}\right)$$
$$+ (\nu_1 + \nu_2 - 1)\ln\left(2 - \frac{x+y}{\sigma}\right),$$

where $x > 0$, $y > 0$ and $x + y < 1$. The second order partial derivatives of $\ln L(\nu_1, \nu_2, \sigma)$ are given by

$$\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_i^2} = \psi_1(\nu_1+\nu_2+1) - \psi_1(\nu_i), \quad i = 1, 2,$$

$$\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma^2} = \frac{(\nu_1+\nu_2)}{\sigma^2} - \frac{2(x+y)/\sigma^3}{1-(x+y)/\sigma} - \left[\frac{(x+y)/\sigma^2}{1-(x+y)/\sigma}\right]^2$$
$$- (\nu_1+\nu_2-1)\left[\frac{2(x+y)/\sigma^3}{2-(x+y)/\sigma} + \left[\frac{(x+y)/\sigma^2}{2-(x+y)/\sigma}\right]^2\right],$$

$$\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_2\,\partial \nu_1} = \psi_1(\nu_1+\nu_2+\sigma),$$

$$\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma\,\partial \nu_i} = -\frac{1}{\sigma} + \frac{(x+y)/\sigma^2}{2-(x+y)/\sigma}, \quad i = 1, 2,$$

where ψ_1 is the polygamma function defined by $\psi_1(z) = \frac{d^2}{dz^2}\ln\Gamma(z)$. Now, noting that the expected value of a constant is the constant itself and using results given in Section 4, we get

$$\mathrm{E}\left[\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_i^2}\right] = \psi_1(\nu_1+\nu_2+1) - \psi_1(\nu_i), \quad i = 1, 2,$$

$$\mathrm{E}\left[\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \nu_2\,\partial \nu_1}\right] = \psi_1(\nu_1+\nu_2+\sigma),$$

$$\mathrm{E}\left[\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma\,\partial \nu_i}\right] = \frac{1}{\sigma}\left[\frac{2}{\nu-1} - \frac{\sqrt{\pi}\nu\Gamma(\nu-1)}{\Gamma(\nu+1/2)}\right], \quad i = 1, 2,$$

$$\mathrm{E}\left[\frac{\partial^2 \ln L(\nu_1,\nu_2,\sigma)}{\partial \sigma^2}\right] = -\frac{1}{\sigma^2}\left[\nu - \frac{3\nu^2+3\nu-2}{(\nu-1)(\nu-2)} - \frac{2\sqrt{\pi}\nu(\nu+2)\Gamma(\nu-1)}{\Gamma(\nu+1/2)}\right]$$
$$- \frac{(\nu-1)}{\sigma^2}\left[2\frac{\sqrt{\pi}\,\Gamma(\nu+1)}{\Gamma(\nu+1/2)} - 2\right.$$
$$\left.+ \sum_{k=0}^{\infty}\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^j\frac{\Gamma(1+j/2)\Gamma(\nu+1)}{\Gamma(\nu+1+j/2)}\right].$$

6 Conclusion

We have defined a bivariate generalization of the Topp-Leone distribution. By using standard definitions and results, several properties of this distribution including marginal distributions, joint and marginal moments, correlation coefficient, and Fisher information matrix have been derived. The exact distributions of $X+Y$, $X/(X+Y)$ and XY when X and Y follow the bivariate Topp-Leone distribution have also been obtained by using transformation of variables. Entropies and estimation of parameters associated with the bivariate Topp-Leone distribution can also be obtained.

The multivariate generalization of the Topp-Leone distribution is also a captivating topic to investigate. The multivariate Topp-Leone distribution with parameters $\nu_1 > 0, \ldots, \nu_n > 0$ and $\sigma > 0$ can be defined by the density

$$\frac{2\Gamma(\nu_1+\cdots+\nu_n+1)}{\sigma^{\nu_1+\cdots+\nu_n}\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\prod_{i=1}^{n} x_i^{\nu_i-1}\left(1-\frac{\sum_{i=1}^n x_i}{\sigma}\right)\left(2-\frac{\sum_{i=1}^n x_i}{\sigma}\right)^{\sum_{i=1}^n \nu_i-1},$$

where $x_i > 0$, $i=1,\ldots,n$ and $\sum_{i=1}^n x_i < \sigma$.

Appendix

The integral representation of the Gauss hypergeometric function is given as

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t, \tag{A.1}$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $|\arg(1-z)| < \pi$. Note that, by expanding $(1-zt)^{-b}$, $|zt|<1$, in (A.1) and integrating t the series expansion for F can be obtained. For properties and results the reader is referred to Luke [16].

Let f be a continuous function and $\mathrm{Re}(\alpha_i) > 0$, $i=1,2$. Then

$$\begin{aligned} D_2(\alpha_1,\alpha_2;f) &= \int_0^1\int_0^{1-x_1} x_1^{\alpha_1-1}x_1^{\alpha_2-1}f(x_1+x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\int_0^1 z^{\alpha_1+\alpha_2-1}f(z)\,\mathrm{d}z, \end{aligned} \tag{A.2}$$

where the second line has been obtained by substituting $y = x_1/z$ and $z = x_1+x_2$ with the Jacobian $J(x_1,x_2\to y,z) = z$ and the beta integral. The integral in (A.2) is a special case of Liouville-Dirichlet integral.

Finally, we define the beta type 1 and beta type 2 distributions. These definitions can be found in Johnson, Kotz and Balakrishnan [11].

Definition A.1. *The random variable X is said to have a beta type 1 distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim \mathrm{B1}(a, b)$, if its pdf is given by $\{\Gamma(a+b)/\Gamma(a)\Gamma(b)\}x^{a-1}(1-x)^{b-1}$, $0 < x < 1$.*

Definition A.2. *The random variable X is said to have a beta type 2 (inverted beta) distribution with parameters (a, b), denoted as $X \sim \mathrm{B2}(a, b)$, if its pdf is given by $\{\Gamma(a+b)/\Gamma(a)\Gamma(b)\}x^{a-1}(1+x)^{-(a+b)}$, $x > 0$, $a > 0$, $b > 0$.*

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