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Fixed-Point Theorems in Fuzzy Metric Spaces

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Abstract

Banach's fixed point theorem gives a general criterion for the iteration procedure of a function to yield a fixed point. As a result, many researchers introduced several fixed point theorems for a given function under different conditions and spaces. The main objective of this study is to present and investigate two fixed point theorems related to two different contractive mappings in addition to some theoretical results concerning fuzzy metric spaces. We also explore heir relationship with nonfuzzy or crisp metric spaces. The fixed point theorems are based on the distance function between fuzzy points.

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1 Introduction

The subject of fuzzy metric space can be considered a simple function in the study of fuzzy sets. Thus, several different definitions are given in relation to this subject using different approaches. Some of these definitions suffer from the inaccuracy of their composition, in which some definitions of the distance function depend on the description of the α-level sets. Others consider the distance function as a fuzzy mapping of a certain membership function, which can be defined using the popular extension principle in fuzzy set theory [7, 18]. The topology properties of fuzzy sets were first introduced and studied by Chang in 1968 [3]. Erceg investigated fuzzy metric spaces in the theory of fuzzy sets in 1979 [4] and their interaction with statistical metric spaces, whilst Zike explored fuzzy points in 1982 [19] and discussed metric spaces with certain metrics defined between the two fuzzy points. The topic of fuzzy metric spaces can be viewed as a fundamental factor in the study of fuzzy sets. Therefore, many different definitions using different approaches are given concerning this subject. Some of these definitions suffer from the inexactness in their structure, in which some definitions of the distance function depend on the definition of the α-level sets and others consider the distance function as a fuzzy mapping with certain membership function that may be defined using the popular extension principle in fuzzy set theory,[14]-[17]. In 1973, Nazaroff offered systematic analysis of abstract fuzzy dynamical structures and Warren provided further results concerning this study. However, Butnariu in 1982 [2] studied the fixed-point theorem in fuzzy mapping and explained some fixed-point theorems with algorithms for fuzzy mappings to estimate certain fixed points when fuzzy mappings have certain defined properties. A study of fuzzy fixed-point theorem using the Hausdorff distance function between the α-level sets corresponding to fuzzy sets was introduced and presented by Fadhel in 1998 [6].

George and Sapena [9] gave the fixed-point theorem according to the sense of George and Vermani [8] for complete fuzzy metric space and also for fuzzy metric spaces of Kramsil, which are complete in the sense of Grabiec, [8]- [10]. In 2012, Fadhel and Majeed [5] investigated the completeness and the fixed-point theorem of a certain fuzzy metric space, which is the family of all fuzzy sets in which any intersection or union of fuzzy sets are fuzzy numbers or fuzzy point. Rana et al. in 2015 [17] explored the contraction mapping principle in fuzzy domain normed spaces of formal balls and also proved the Banach fixed point theorem in this space. Patir et al. in 2018 derived and proven some fixed-point theorem in fuzzy metric spaces for selfmappings with different types of contractive conditions [16]. An extension of the generalised Carist's fixed point theorem proven by Bollenbacher and Hicks for p-orbitally complete fuzzy metric spaces was given by Karayilan and Telci in 2019 [13]. Huang introduced in 2021 new types of contraction mapping, which are called f-contraction; some fixed-point theorems were introduced for such types of contraction [11]. Zainb et al. established in 2020 a method for generalised weakly contraction mapping of the fixed-point theory [20]. Khan et al. in 2021 established sequential characterisation properties of Lebesgue metric space by using contraction mapping in complete metric space [14].

The essential objective of this study is to present and analyse two fixed point theorems corresponding with two contractive mappings in addition to introducing some analytical results related to fuzzy metric spaces. Their relationship with nonfuzzy or crisp metric spaces is also explored. The fixed-point theorems are based on the distance function defined between fuzzy points.

2 Preliminaries and Basic Concepts

The abbreviations and the basic and fundamental concepts in fuzzy metric spaces used in this study are defined as follows; X is the universal nonempty set, I is the unit interval [0,1], I^X is the set of all fuzzy subsets of X, (X, d_X) is used to denote the nonfuzzy (or crisp) metric space, (I^X, d_{I^X}) is the corresponding fuzzy metric space and p_x^α is the fuzzy point with support set to be consisted of only one element $x \in X$ and membership function to be equal to $\alpha \in [0, 1]$. The complement of p_x^α is also a fuzzy point, which is denoted by $p_x^{1-\alpha}$. The fuzzy point p_x^α is also a fuzzy set, and therefore, I^X will be the set of all fuzzy points in X.

The fuzzy distance function for constructing fuzzy metric spaces is defined and introduced here as a nonfuzzy function defined between two fuzzy points.

Definition 2.1. [19]. *Let (X, d_X) be a metric space, then the mapping $d_{I^X} : I^X \times I^X \longrightarrow \mathbb{R}^+$ is a distance function between fuzzy points if for every fuzzy points $p_x^\alpha, p_y^\beta, p_z^\gamma \in I^X$, where $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0, 1]$; the mapping d_{I^X} fulfills the following conditions:*

1. $d_{I^X}(p_x^\alpha, p_y^\beta) = 0$ *if and only if* $x = y$ *and* $\alpha = \beta$.

2. $d_{I^X}(p_x^\alpha, p_y^\beta) = d_{I^X}(p_y^{1-\beta}, p_x^{1-\alpha})$.

3. $d_{I^X}(p_x^\alpha, p_y^\beta) \le d_{I^X}(p_x^\alpha, p_z^\gamma) + d_{I^X}(p_z^\gamma, p_y^\beta)$.

4. $d_{I^X}(p_x^\alpha, p_y^\beta) \le r$, *where* $r \ge 0$, *implies that there exists* $\hat{\alpha} > \alpha$ *such that* $d_{I^X}(p_x^{\hat{\alpha}}, p_y^\beta) < r$.

The pair (I^X, d_{I^X}) *is called fuzzy metric space.*

The definition of the fuzzy distance function that will be used in the following work is also provided in the next remark in which its proof is straightforward based on Definition 2.1.

Remark 2.2. *Assume* (X, d_X) *is a nonfuzzy metric space and* (I^X, d_{I^X}) *is the corresponding fuzzy metric space; the distance between two fuzzy points may be defined as:*

$$d_{I^X}(p_x^\alpha, p_y^\beta) = |\alpha - \beta| + d_X(x, y), \text{ for all } x, y \in X \text{ and } \alpha, \beta \in [0, 1] \quad (2.1)$$

It is notable that, also other examples of fuzzy distance functions may be achieved depending on the definition of the nonfuzzy distance function $d_X(x, y)$ given in equation (2.1).

3 Completeness of Fuzzy Metric Spaces

This section provides the essential conditions for stating and proving the fuzzy fixed-point theorem, which is often known as the Banach fixed point theorem. It is the completeness of fuzzy metric space and its relationship with the completeness of the crisp metric space. Therefore, some of the basic properties of these spaces that are inherited between them are studied for the following work (additional details may be found in [1, 7, 10, 12]). The definition of the convergence of a sequence of fuzzy points is provided first and then the definition of Cauchy sequence of fuzzy points is given.

Definition 3.1. [10]. *In* I^X, *a fuzzy points sequence* $< p_{x_n}^{\alpha_n} >$, $n \in \mathbb{N}$ *is said to be converge to a fuzzy point* $p_x^\alpha \in I^X$ *(which is denoted by* $p_{x_n}^{\alpha_n} \longrightarrow p_x^\alpha$*) if for every* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$, *such that:*

$$d_I^X(p_{x_n}^{\alpha_n}, p_x^\alpha) < \varepsilon, for\ all\ n \ge N$$

where $p_{x_n}^{\alpha_n},\ p_x^\alpha \in I^X,\ x_n, x \in X$ *and* $\alpha_n,\ \alpha \in [0, 1]$.

Definition 3.2. [10]. *A sequence of fuzzy points* $< p_{x_n}^{\alpha_n} >$, $n \in \mathbb{N}$ *in* I^X *is said to be Cauchy sequence of fuzzy points if for every* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$, *such that:*

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) < \varepsilon, \; for \; all \; n, \; m \geq N$$

where $p_{x_n}^{\alpha_n}$, $p_{x_m}^{\alpha_m} \in I^X$, x_n, $x_m \in X$ *and* α_n, $\alpha_m \in [0,1]$.

The next remark may be achieved and proved easily relative to the fuzzy distance function given in equation (2.1).

Remark 3.3. *A sequence* $< p_{x_n}^{\alpha_n} >$, x_n, $\alpha_n \in [0,1]$, $n \in \mathbb{N}$ *of fuzzy points is converge to a fuzzy point* p_x^α *if and only if there exists a crisp monotonic sequence of real numbers* $< \alpha_n >$, $n \in \mathbb{N}$ *in* $[0,1]$ *converge to* $\alpha \in \mathbb{R}$ *and a sequence of support points* $< x_n >$, $x_n \in X$, $n \in \mathbb{N}$, *which is converge to* $x \in X$, *as* n *tends to* ∞.

Also, the following theorem describes the relation between convergent sequences of fuzzy points and Cauchy sequences of fuzzy points.

Theorem 3.4. *Let* $< p_{x_n}^{\alpha_n} >$, $x_n \in X$, $\alpha_n \in [0,1]$, $n \in \mathbb{N}$ *be a convergent sequence of fuzzy points, then* $< p_{x_n}^{\alpha_n} >$ *is also a Cauchy sequence of fuzzy points.*

Proof. Suppose that $< p_{x_n}^{\alpha_n} >$ is a convergent sequence of fuzzy points, hence from Definition 3.1, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $d_{I^X}(p_{x_n}^{\alpha_n}, p_x^\alpha) < \varepsilon$, for all $n \geq N$.
In order to demonstrate that $< p_{x_n}^{\alpha_n} >$ is a Cauchy sequence of fuzzy points, it must be proved that $d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) < \varepsilon$.
Now, using the triangle inequality (Definition 2.1(3)), it is induced that:

$$\begin{aligned} d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) &< d_{I^X}(p_{x_n}^{\alpha_n}, p_x^\alpha) + d_{I^X}(p_x^\alpha, p_{x_m}^{\alpha_m}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $< p_{x_n}^{\alpha_n} >$ is a Cauchy sequence of fuzzy points. □

Using Remark 3.3 and Theorem 3.4, the next corollary may be simply stated and proved:

Corollary 3.5. *Let* (I^X, d_{I^X}) *be a fuzzy metric space, then* $< p_{x_n}^{\alpha_n} >$, $x_n \in X$, $\alpha_n \in [0,1]$, $n \in \mathbb{N}$ *is a Cauchy sequence of fuzzy points if and only if two nonfuzzy Cauchy sequences exist, namely the support points Cauchy sequence* $< x_n > \in X$ *and a monotonic real numbers Cauchy sequence* $< \alpha_n > \in [0,1]$.

The completeness of fuzzy metric space may be defined likewise as that given in nonfuzzy metric spaces, except using the fuzzy points, as it is given next:

Definition 3.6. *A fuzzy metric space (I^X, d_{I^X}) is said to be complete fuzzy metric space, if every Cauchy sequence of fuzzy points in I^X converges to a fuzzy point in I^X.*

The relationship between the completeness of the nonfuzzy metric space (X, d) and the completeness of the fuzzy metric space (I^X, d_{I^X}) is one of the most important results obtained in this paper.

Theorem 3.7. *Suppose that (I^X, d_{I^X}) is the fuzzy metric space induced from the nonfuzzy or crisp metric space (X, d_X), then (I^X, d_{I^X}) is complete if and only if (X, d_X) is a complete.*

Proof. Suppose that (X, d_X) is a complete nonfuzzy metric space and to prove that (I^X, d_{I^X}) is a complete fuzzy metric space
Let $< p_{x_n}^{\alpha_n} >$, $x_n \in X$, $n \in \mathbb{N}$, $\alpha \in [0,1]$ be a Cauchy sequence of fuzzy points in (I^X, d_{I^X}) and to prove that it is converge to a fuzzy point $p_x^\alpha \in I^X$.
By Definition 3.2, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, so that:

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) < \varepsilon, \; for\, all\; n, \, m \geq N \tag{3.2}$$

and from Corollary 3.5 it is implied that inequality (3.2) is equivalent to the existence of two nonfuzzy Cauchy sequences, namely a sequence of supports $< x_n > \in X$ and an α-levels sequence of real numbers $< \alpha_n > \in [0,1]$, such that:

$$d_X(x_n, x_m) < \varepsilon \; and \; |\alpha_n - \alpha_m| < \varepsilon, \; for\, all\; n, m \geq N$$

To prove that $< x_n >$ and $< \alpha_n >$ converges to x and α, respectively
Since $< x_n >$ is a Cauchy sequence of support points in (X, d_X), which is a complete nonfuzzy metric space and hence $< x_n >$ convergent to $x \in X$. Moreover, $< \alpha_n >$ is a Cauchy sequence of real numbers in [0,1], therefore from the completeness of the field of real numbers implies that $< \alpha_n >$ converge to $\alpha \in [0,1]$.
Therefore from Remark 3.3, implies that $d_{I^X}(p_{x_n}^{\alpha_n}, p_x^\alpha) < \varepsilon$ for all $n \geq N$, i.e., $< p_{x_n}^{\alpha_n} >$ is a convergent sequence in (I^X, d_{I^X})
Thus (I^X, d_{I^X}) is a complete fuzzy metric space.

The converse direction may be proved easily. □

4 Contractive Fuzzy Mappings and Banach Fuzzy Fixed-Point Theorems

In this section, the contractive mapping theory in fuzzy metric spaces is studied before stating and proving the Banach fixed point theorem in fuzzy metric spaces. Many definitions of contractive mappings exist, and we introduce only two definitions of them. The first contractive mapping definition is the usual definition, whilst the second contractive mapping definition is the generalisation of the first one.

Definition 4.1. *Let (I^X, d_{I^X}) be a fuzzy metric space and $f : I^X \longrightarrow I^X$ be a fuzzy mapping, then f is a contractive fuzzy mapping in I^X if there exists $k \in (0,1)$, such that:*

$$d_{I^X}(f(p_x^\alpha), f(p_y^\beta)) \leq k d_{I^X}(p_x^\alpha, p_y^\beta) \tag{4.3}$$

For all fuzzy points p_x^α, $p_y^\beta \in I^X$ and α, $\beta \in [0,1]$.

The first fixed point theorem of this work is given in the next theorem:

Theorem 4.2. *Let (I^X, d_{I^X}) be a complete fuzzy metric space and $f : I^X \longrightarrow I^X$ be a contractive fuzzy mapping, then f has a fuzzy fixed point in I^X.*

Proof. If $p_{x_0}^{\alpha_0} \in I^X$, and suppose that $p_{x_1}^{\alpha_1} = f(p_{x_0}^{\alpha_0})$, $p_{x_2}^{\alpha_2} = f(p_{x_1}^{\alpha_1}) = f(f(p_{x_0}^{\alpha_0})) = f^2(p_{x_0}^{\alpha_0})$, $p_{x_3}^{\alpha_3} = f(p_{x_2}^{\alpha_2}) = f^3(p_{x_0}^{\alpha_0})$, $\ldots$, $p_{x_n}^{\alpha_n} = f^n(p_{x_0}^{\alpha_0})$, where f^n refers to n^{th}-composition of the fuzzy mapping f.
Now, let $< p_{x_n}^{\alpha_n} >$, $x_n \in X$, $\alpha_n \in [0,1]$, $n \in \mathbb{N}$, be a sequence of fuzzy points in I^X and from the contractility of f and using inequality (4.3), the following are obtained:

$$\begin{aligned}
d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_2}^{\alpha_2}) &= d_{I^X}(f(p_{x_0}^{\alpha_0}), f(p_{x_1}^{\alpha_1})) \\
&\leq k d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1}) \\
d_{I^X}(p_{x_2}^{\alpha_2}, p_{x_1}^{\alpha_1}) &= d_{I^X}(f(p_{x_1}^{\alpha_1}), f(p_{x_2}^{\alpha_2})) \\
&\leq k d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_2}^{\alpha_2}) \\
&\leq k^2 d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1}) \\
&\vdots \\
d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_{n-1}}^{\alpha_{n-1}}) &\leq k^n d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1})
\end{aligned}$$

Also, the triangle inequality of Definition 2.1 is used to get:

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) \leq d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_k}^{\alpha_k}) + d_{I^X}(p_{x_k}^{\alpha_k}, p_{x_m}^{\alpha_m})$$

Therefore, application of equation (2.1), will give:

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) = |\alpha_n - \alpha_m| + d(x_n, x_m)$$

and hence, for each nonfuzzy sequence $< x_n >$ and $< \alpha_n >$ and by using the triangle inequality, give:

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$$

and

$$|\alpha_n - \alpha_{n+2}| \leq |\alpha_n - \alpha_{n+1}| + |\alpha_{n+1} - \alpha_{n+2}|$$

Thus, carrying out some calculations will result the following:

$$\begin{aligned} d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) &\leq d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_{n+1}}^{\alpha_{n+1}}) + d_{I^X}(p_{x_{n+1}}^{\alpha_{n+1}}, p_{x_{n+2}}^{\alpha_{n+2}}) + \ldots + d_{I^X}(p_{x_{m-1}}^{\alpha_{m-1}}, p_{x_m}^{\alpha_m}) \\ &\leq (k^n + k^{n+1} + ... + k^{m-1}) d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1}) \\ &= k^n(1 + k + ... + k^{m-1-n} + ...) d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1}) \\ &= \frac{k^n}{1-k} d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1}), \ \alpha \in [0, 1] \end{aligned}$$

Hence, for any $\varepsilon > 0$, let N be a natural number, which is large enough to guarantee that:

$$\frac{k^n}{1-k} d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_1}^{\alpha_1}) < \varepsilon$$

Therefore, for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $n, m \geq N$, so that $d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) \leq \varepsilon$
Hence, $< p_{x_n}^{\alpha_n} >$, $n \in \mathbb{N}$ is a Cauchy sequence of fuzzy points in I^X
Since (I^X, d_{I^X}) is a complete fuzzy metric space, then $< p_{x_n}^{\alpha_n} >$ converge to a fuzzy point $p_x^\alpha \in I^X$

Now, to show that p_x^α is a really the fixed point of f. Since $p_{x_n}^{\alpha_n} \longrightarrow p_x^\alpha$ and f is continuous, then $f(p_{x_n}^{\alpha_n}) \longrightarrow f(p_x^\alpha)$, i.e., $p_{x_{n+1}}^{\alpha_{n+1}} \longrightarrow f(p_x^\alpha)$. Also, since $p_{x_n}^{\alpha_n} \longrightarrow p_x^\alpha$, then $f(p_{x_n}^{\alpha_n}) = p_x^\alpha$, and so p_x^α is a fixed point of this mapping. □

Following the above, the second definition of contractive mapping considered in this work for fuzzy metric spaces, which will be introduced and used to establish the second main result of this article, which is also a fuzzy fixed-point theorem.

Definition 4.3. *Let (I^X, d_{I^X}) be a complete fuzzy metric space, a fuzzy mapping $f : I^X \longrightarrow I^X$ is called contractive on I^X if there exists $h \in (0,1)$, such that for all p_x^α, $p_y^\beta \in I^X$, where $x, y \in X$ and $\alpha, \beta \in [0,1]$ implies to:*

$$d_{I^X}(f(p_x^\alpha), f(p_y^\beta)) \leq h \operatorname{Max}\left\{d_{I^X}(p_x^\alpha, f(p_x^\alpha)), d_{I^X}(p_y^\beta, f(p_y^\beta))\right\} \tag{4.4}$$

Next, we demonstrate the statements of the second fuzzy fixed-point theorem in terms of Definition 4.3 and thereafter introduce its proof.

Theorem 4.4. *Given a complete fuzzy metric space (I^X, d_{I^X}) and a contractive fuzzy mapping $f : I^X \longrightarrow I^X$ satisfying inequality (4.4), then f has a fuzzy fixed point in I^X.*

Proof. Since:

$$d_{I^X}(f(p_x^\alpha), f(p_y^\beta)) \leq h \operatorname{Max}\left\{d_{I^X}(p_x^\alpha, f(p_x^\alpha)), d_{I^X}(p_y^\beta, f(p_y^\beta))\right\}, \ h \in (0,1)$$

Then by letting $p_{x_0}^{\alpha_0} \in I^X$ and suppose that $p_{x_1}^{\alpha_1} = f(p_{x_0}^{\alpha_0})$, $p_{x_2}^{\alpha_2} = f(p_{x_1}^{\alpha_1}) = f(f(p_{x_0}^{\alpha_0}) = f^2(p_{x_0}^{\alpha_0})$, $p_{x_3}^{\alpha_3} = f(p_{x_2}^{\alpha_2}) = f(f^2(p_{x_0}^{\alpha_0})) = f^3(p_{x_0}^{\alpha_0})$, $\dots$, $p_{x_n}^{\alpha_n} = f^n(p_{x_0}^{\alpha_0})$
Now:

$$\begin{aligned}
d_{I^X}(p_{x_2}^{\alpha_2}, p_{x_3}^{\alpha_3}) &= d_{I^X}(f(p_{x_1}^{\alpha_1}), f(p_{x_2}^{\alpha_2})) \\
&\leq h \operatorname{Max}\left\{d_{I^X}(p_{x_1}^{\alpha_1}, f(p_{x_1}^{\alpha_1})), d_{I^X}(p_{x_2}^{\alpha_2}, f(p_{x_2}^{\alpha_2}))\right\} \\
&= h \operatorname{Max}\left\{d_{I^X}(f(p_{x_0}^{\alpha_0}), f(p_{x_1}^{\alpha_1})), d_{I^X}(f(p_{x_1}^{\alpha_1}), f(p_{x_2}^{\alpha_2}))\right\} \\
&\leq h \operatorname{Max}\left\{h \operatorname{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, f(p_{x_0}^{\alpha_0})), d_{I^X}(p_{x_1}^{\alpha_1}, f(p_{x_1}^{\alpha_1}))\right\},\right. \\
&\quad \left. h \operatorname{Max}\left\{d_{I^X}(p_{x_1}^{\alpha_1}, f(p_{x_1}^{\alpha_1})), d_{I^X}(p_{x_2}^{\alpha_2}, f(p_{x_2}^{\alpha_2}))\right\}\right\} \\
&= h^2 \operatorname{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, f(p_{x_0}^{\alpha_0})), d_{I^X}(p_{x_1}^{\alpha_1}, f(p_{x_1}^{\alpha_1})), d_{I^X}(p_{x_2}^{\alpha_2}, f(p_{x_2}^{\alpha_2}))\right\}
\end{aligned}$$

Similarly:

$$\begin{aligned}
d_{I^X}(p_{x_3}^{\alpha_3}, p_{x_4}^{\alpha_4}) \leq h^3 \operatorname{Max}\{&d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_1}^{\alpha_1}), d_{I^X}(p_{x_2}^{\alpha_2}, p_{x_2}^{\alpha_2}), \\
&d_{I^X}(p_{x_3}^{\alpha_3}, p_{x_3}^{\alpha_3})\}
\end{aligned}$$

$$\vdots$$

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_{n+1}}^{\alpha_{n+1}}) \leq h^n \operatorname{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_1}^{\alpha_1}), \dots, d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_n}^{\alpha_n})\right\}$$

By using the triangle inequality, we have:

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) \leq d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_k}^{\alpha_k}) + d_{I^X}(p_{x_k}^{\alpha_k}, p_{x_m}^{\alpha_m}), \text{for all } m \geq n$$

Hence:

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) \leq d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_{n+1}}^{\alpha_{n+1}}) + d_{I^X}(p_{x_{n+1}}^{\alpha_{n+1}}, p_{x_{n+2}}^{\alpha_{n+2}}) + \ldots + d_{I^X}(p_{x_{m-1}}^{\alpha_{m-1}}, p_{x_m}^{\alpha_m})$$

$$\begin{aligned} &\leq h^n \text{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), \ldots, d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_n}^{\alpha_n})\right\} + \\ &h^{n+1}\text{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), ..., d_{I^X}(p_{x_{n+1}}^{\alpha_{n+1}}, p_{x_{n+1}}^{\alpha_{n+1}})\right\} + ...+ \\ &h^{m-1}\text{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), \ldots, d_{I^X}(p_{x_{m-1}}^{\alpha_{m-1}}, p_{x_{m-1}}^{\alpha_{m-1}})\right\} \end{aligned}$$

$$\begin{aligned} \leq h^n \text{Max}&\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), ..., d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_n}^{\alpha_n}), d_{I^X}(p_{x_{n+1}}^{\alpha_{n+1}}, p_{x_{n+1}}^{\alpha_{n+1}}), ..., \right.\\ &\left. d_{I^X}(p_{x_{m-1}}^{\alpha_{m-1}}, p_{x_{m-1}}^{\alpha_{m-1}})\right\} + h^{n+1}\text{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), ..., \right.\\ &\left. d_{I^X}(p_{x_{n+1}}^{\alpha_{n+1}}, p_{x_{n+1}}^{\alpha_{n+1}}), .., d_{I^X}(p_{x_{m-1}}^{\alpha_{m-1}}, p_{x_{m-1}}^{\alpha_{m-1}})\right\} + ...+ \\ &h^{m-1}\text{Max}\left\{d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}), ..., d_{I^X}(p_{x_{m-1}}^{\alpha_{m-1}}, p_{x_{m-1}}^{\alpha_{m-1}})\right\} \end{aligned} \tag{4.5}$$

Now, if the maximization of inequality (4.5) occurs at $d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0})$, then:

$$\begin{aligned} d_{I^X} p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) &\leq h^n d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) + h^{n+1} d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) + ...+ \\ &\quad h^{m-1} d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) \\ &= (h^n + h^{n+1} + ... + h^{m-1}) d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) \\ &= h^n(1 + h + \ldots + h^{m-1-n}) d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) \\ &\leq h^n(1 + h + \ldots + h^{m-1-n} + \ldots) d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) \\ &= \frac{h^n}{1-h} d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}),\ 0 \leq h < 1 \end{aligned}$$

Hence, for $\varepsilon > 0$, let N be a natural number large enough to guarantee that $\frac{h^n}{1-h} d_{I^X}(p_{x_0}^{\alpha_0}, p_{x_0}^{\alpha_0}) < \varepsilon$, for all $n \geq N$
Therefore $d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) \leq \varepsilon$, for all $m \geq n$
Thus, $< p_{x_n}^{\alpha_n} >$, for all $n \in \mathbb{N}$ is a Cauchy sequence of fuzzy points in I^X and since (I^X, d_{I^X}) is a complete fuzzy metric space, then $< p_{x_n}^{\alpha_n} >$ converges to a fuzzy point $p_x^\alpha \in I^X$
Now, p_x^α is the fixed point of f since $p_{x_n}^{\alpha_n} \longrightarrow p_x^\alpha$ and f is continuous, then $p_{x_n}^{\alpha_n} \longrightarrow f(p_x^\alpha)$, i.e., we may write $p_{x_{n+1}}^{\alpha_{n+1}} \longrightarrow f(p_x^\alpha)$ as $n \longrightarrow \infty$ and we have $f(p_{x_n}^{\alpha_n}) \longrightarrow f(p_x^\alpha)$
Therefore, from the uniqueness of the limit point, we have $f(p_x^\alpha) = p_x^\alpha$.
Similarly, if the maximization of inequality (4.5) occurs at $d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_1}^{\alpha_1})$, we will get:

$$d_{I^X}(p_{x_n}^{\alpha_n}, p_{x_m}^{\alpha_m}) \leq \frac{h^n}{1-h} d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_1}^{\alpha_1}),\ 0 < h < 1$$

and also for $\varepsilon > 0$ let N be a natural number large enough to guarantee that $\frac{h^n}{1-h} d_{I^X}(p_{x_1}^{\alpha_1}, p_{x_1}^{\alpha_1}) < \varepsilon$, for all $n \geq N$

Therefore $d_{I^X}(p^{\alpha_n}_{x_n}, p^{\alpha_m}_{x_m}) \leq \varepsilon$, for all $m \geq n$ and also we get p^{α}_{x} is a fixed point of f
So, on proceeding similarly for $m-1$, we get also p^{α}_{x} is a fixed point of f
This completes the proof of the theorem. □

5 Conclusion

Fixed point theory in fuzzy and nonfuzzy metric spaces are important and have many applications in other fields of mathematics. Contractive mappings have different definitions. Therefore, fixed point theorem may be proven depending on the specific type of the contractive mapping and the application of the theorem.

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