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The Study on the Relative Rank of Transformation Semigroups with Restricted Range

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Abstract

Let X be an infinite set and let Y be a non-empty subset of X. Then $\mathcal{T}(X,Y)$ is the semigroup of all full transformations from set X to set Y which is called the transformation semigroup with restricted range. In this paper, we determine the relative rank $\mathcal{T}(X,Y)$ modulo the top $\mathcal{J}$ class. Moreover, we give examples of the relative rank $\mathcal{T}(X,Y)$ modulo $\mathcal{S}(X,Y)$ and modulo $\mathcal{E}(X,Y)$ to obtain that the relative rank is infinite for a finite set Y.

1 Introduction and Preliminaries

Let S be a semigroup. As usual, the rank of a finite semigroup S is the minimum size or cardinality of a generating set A of S which is denoted by

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$$rank(S) := min\{|A| : A \subseteq S, \langle A\rangle = S\}.$$

In [5], Howie, Ruškuc and Higgins explained the rank of various well known semigroups such as a finite full transformation semigroup has rank 3, a finite partial transformation semigroup has rank 4 and a finite symmetric inverse semigroup has rank 3. For an infinite set S, we have $rank(S) = |S|$ and the notation of rank in this case is not useful. Then the authors in [5] defined a new definition of rank which is called the relative rank of S modulo a subset A of S by

$$rank(S : A) := min\{|B| : B \subseteq S, \langle A \cup B\rangle = S\}.$$

Therefore, we immediately obtain $rank(S : \emptyset) = rank(S)$, $rank(S : S) = 0$, $rank(S : A) = rank(S : \langle A\rangle)$ and $rank(S : A) = 0$ if and only if $\langle A\rangle = S$.

Let X be a nonempty set and let $\mathcal{T}(X)$ be the set of all full transformations from set X into itself. Then $\mathcal{T}(X)$ is a semigroup of all full transformations under the composition of functions. It is easy to see that $|\mathcal{T}(X)| = |X|^{|X|}$ when X is a finite set and $|\mathcal{T}(X)|$ is uncountable when X is an infinite set. For an infinite set X, the relative rank $\mathcal{T}(X)$ modulo $\mathcal{E}(X)$ the set of all idempotents in $\mathcal{T}(X)$ and modulo $\mathcal{S}(X)$ the symmetric group on X are equal to 2 in [5]. Let Y be a non-empty subset of X. Then the semigroup $\mathcal{T}(X, Y)$ was first introduced and studied by Symons in [8] which is denoted by $\mathcal{T}(X, Y)$ and defined by

$$\mathcal{T}(X, Y) := \{\alpha \in \mathcal{T}(X) : X\alpha \subseteq Y\}.$$

Clearly, $\mathcal{T}(X, Y)$ is a subsemigroup of $\mathcal{T}(X)$. If $X = Y$, then $\mathcal{T}(X, Y) = \mathcal{T}(X)$. The semigroup $\mathcal{T}(X, Y)$ is also called a transformation semigroup with restricted range and it was intensively studied in the last two deacade [1], [7] and [9]. Then we can define various parameters asscociated to a transformation semigroup with restricted range as the following definitions.

Definition 1.1. *Let $\alpha \in \mathcal{T}(X, Y)$. Then $rank(\alpha)$ is defined to be the cardinality of image of α; i.e.*

$$rank(\alpha) := |im(\alpha)|.$$

Definition 1.2. *Let $\alpha \in \mathcal{T}(X, Y)$. Then the defect of α is defined by*

$$d(\alpha) := |Y \setminus im(\alpha)|.$$

Consider the kernel relation of α that is defined by

$$\ker(\alpha) := \{(x, y) \in X \times X : x\alpha = y\alpha\}.$$

It is easy to verify that $\ker(\alpha)$ is an equivalence relation (reflexive, symmetric, and transitive) on X. For convinience, we write $B \in \ker(\alpha)$ which means $B \subseteq X$ and it is called a block of $\ker(\alpha)$. A set T_α with $|B \cap T_\alpha| = 1$ for all blocks B of $\ker(\alpha)$. Then it is called a transversal of $\ker(\alpha)$. So, we can define an infinite contraction index of $\alpha \in \mathcal{T}(X, Y)$ as the following definition.

Definition 1.3. *Let $\alpha \in \mathcal{T}(X, Y)$. Then the infinite contraction index of α is defined as the number of blocks of $\ker(\alpha)$ of size $|X|$.*

In other words, if we let

$$K(\alpha) := \{x \in im(\alpha) : |x\alpha^{-1}| = |X|\},$$

then the infinite contraction index is

$$k(\alpha) := |K(\alpha)|.$$

Let us recall singular and regular cardinals in [3]. A cardinal κ is singular if there exist sets Y and Z_y $(y \in Y)$ such that

$$|Y| < \kappa, |Z_y| < \kappa, (y \in Y),$$

but

$$|\bigcup_{y \in Y} Z_y| = \kappa.$$

A cardinal κ is regular if it is not singular. Then we obtain that the concepts given in Definition 1.1-1.3 coincide with these for transformations in $\mathcal{T}(X)$ for the case $X = Y$.

In this paper, we determine the relative rank $\mathcal{T}(X, Y)$ modulo the top $\mathcal{J}$ class. We also describe the relative rank $\mathcal{T}(X, Y)$ modulo $\mathcal{S}(X, Y)$ the set of all extentions of the bijections on Y and modulo $\mathcal{E}(X, Y)$ the set of all idempotents in $\mathcal{T}(X, Y)$. In addition, we also give examples of the relative rank $\mathcal{T}(X, Y)$ modulo $\mathcal{S}(X, Y)$ and modulo $\mathcal{E}(X, Y)$ to obtain that the relative rank are infinite for a finite set Y.

2 Results

2.1 Relative rank of $\mathcal{T}(X,Y)$ modulo the top $\mathcal{J}$ class

In this section, we give the properties of the defect and the infinite contraction index of a transformation in $\mathcal{T}(X,Y)$. In particular, we also determine the relative rank of $\mathcal{T}(X,Y)$ modulo the top $\mathcal{J}$ class which is denoted by $\mathcal{J}_1$ and defined by

$$\mathcal{J}_1 := \{\alpha \in \mathcal{T}(X,Y) \ : \ rank(\alpha) = |Y|\}.$$

Proposition 2.1. *Let X be an infinite set and let Y be a subset of X with $|Y| = |X|$. If $\alpha, \beta \in \mathcal{T}(X,Y)$, then we obtain the following properties :*

1. *$d(\alpha\beta) \leq d(\alpha) + d(\beta)$;*
2. *If $|X|$ is a regular cardinal then $k(\alpha\beta) \leq k(\alpha) + k(\beta)$.*

Proof. 1. Since we know that $Y \setminus X\alpha\beta = (Y \setminus X\beta) \cup (X\beta \setminus X\alpha\beta) \subseteq (Y \setminus X\beta) \cup (Y \setminus X\alpha)\beta$.
By the definition of defect, we obtain that

$$\begin{aligned} d(\alpha\beta) = |Y \setminus X\alpha\beta| &\leq |Y \setminus X\beta| + |(Y \setminus X\alpha)\beta| \\ &\leq |Y \setminus X\beta| + |Y \setminus X\alpha| = d(\alpha) + d(\beta). \end{aligned}$$

Hence, we have $d(\alpha\beta) \leq d(\alpha) + d(\beta)$.

2. Assume that $x \in K(\alpha\beta)$. By the definition, we have

$$|X| = |x(\alpha\beta)^{-1}| = |x\beta^{-1}\alpha^{-1}| = |\bigcup_{y \in x\beta^{-1}} y\alpha^{-1}|.$$

Since $|X|$ is a regular cardinal, then $|x\beta^{-1}| = |X|$ or $|y\alpha^{-1}| = |X|$ for some $y \in x\beta^{-1}$. Then we consider two cases:

Case 1. $|x\beta^{-1}| = |X|$. Then we have $x \in K(\beta)$;

Case 2. $|y\alpha^{-1}| = |X|$ for some $y \in x\beta^{-1}$. So, we have $y \in K(\alpha)$. Thus $y\beta \in K(\alpha)\beta$ that is $x \in K(\alpha)\beta$.

Therefore, we can conclude that if $x \in K(\alpha\beta)$, then $x \in K(\beta)$ or $x \in K(\alpha)\beta$. Hence, $K(\alpha\beta) \subseteq K(\beta) \cup K(\alpha)\beta$. Then $k(\alpha\beta) = |K(\alpha\beta)| \leq |K(\alpha)\beta| + |K(\beta)| \leq |K(\alpha)| + |K(\beta)| = k(\alpha) + k(\beta)$. □

Theorem 2.2. *Let X be an infinite set with a regular cardinality and let Y be a non-empty subset of X. Then $\langle \mathcal{J}_1 \rangle = \mathcal{T}(X,Y)$.*

Proof. We want to show that $\langle \mathcal{J}_1 \rangle = \mathcal{T}(X,Y)$. Clearly, $\langle \mathcal{J}_1 \rangle \subseteq \mathcal{T}(X,Y)$. Then we shall prove that $\mathcal{T}(X,Y) \subseteq \langle \mathcal{J}_1 \rangle$. So, we will consider two cases:

Case 1. $|Y| = |X|$. Since $|Y| = |X|$, there is a bijection $\beta : X \to Y$. Let $\alpha \in \mathcal{T}(X,Y)$. Then $\alpha\beta^{-1} \in \mathcal{T}(X)$. By Theorem 3.1 in [5], there are $\beta_1, \beta_2, ..., \beta_k \in \mathcal{T}(X)$ with $rank(\beta_1) = rank(\beta_2) = \cdots = rank(\beta_k) = |Y|$ such that $\alpha\beta^{-1} = \beta_1\beta_2...\beta_k$. For $i = 2, 3, ..., k$, let $\gamma_i \in \mathcal{T}(X,Y)$ be any mapping extending $\beta^{-1}\beta_i\beta$. Clearly, $\gamma_i \in \mathcal{J}_1$ for $i = 2, 3, ..., k$ as well as $\beta_1\beta \in \mathcal{J}_1$. Then we have

$$\alpha = \alpha\beta^{-1}\beta = \beta_1\beta_2...\beta_k\beta = \beta_1\beta\beta^{-1}\beta_2\beta\beta^{-1}...\beta\beta^{-1}\beta_k\beta = \beta_1\beta\gamma_2\gamma_3...\gamma_k \in \langle \mathcal{J}_1 \rangle.$$

Case 2. $|Y| < |X|$. Let $\alpha \in \mathcal{T}(X,Y)$. Then there is $z \in im(\alpha) \subseteq Y$ and $|z\alpha^{-1}| = |X|$ because $|X|$ is regular.

Let δ be a surjection from $z\alpha^{-1}$ to $(Y \setminus im(\alpha)) \cup \{z\}$. Then we define a mapping $\gamma : X \to Y$ by

$$x\gamma = \begin{cases} x\alpha & ; \quad x \in X \setminus z\alpha^{-1} \\ x\delta & ; \quad x \in z\alpha^{-1}. \end{cases}$$

Clearly, $im(\gamma) = Y$, that is $rank(\gamma) = |Y|$. Then we have $\gamma \in \mathcal{J}_1$. Let ϕ be an injection from $(Y \setminus im(\alpha)) \cup \{z\}$ to $X \setminus Y$. So, we define a mapping $\beta : X \to Y$ by

$$x\beta = \begin{cases} x & ; \quad x \in im(\alpha) \\ x\phi^{-1} & ; \quad x \in \phi((Y \setminus im(\alpha)) \cup \{z\}) \\ z & ; \quad otherwise. \end{cases}$$

Clearly, $\beta \in \mathcal{T}(X,Y)$ with $rank(\beta) = |Y|$ and so $\beta \in \mathcal{J}_1$. Let $x \in X$. Therfore, we consider two cases:

Case 2.1. $x \in X \setminus z\alpha^{-1}$. Then $x\gamma\beta = (x\alpha)\beta = x\alpha$.

Case 2.2. $x \in z\alpha^{-1}$. Then $x\gamma\beta = (x\delta)\beta = z = x\alpha$.

Altogether, we have $\alpha = \gamma\beta \in \langle \mathcal{J}_1 \rangle$, i.e. $\mathcal{T}(X,Y) \subseteq \langle \mathcal{J}_1 \rangle$ and so $\langle \mathcal{J}_1 \rangle = \mathcal{T}(X,Y)$. □

From Theorem 2.2, we obtain immediately the relative rank of $\mathcal{T}(X,Y)$ modulo the top $\mathcal{J}$ class as the following corollary.

Corollary 2.3. $rank(\mathcal{T}(X,Y) : \mathcal{J}_1) = 0$.

Proof. By Theorem 2.2, we have $\langle \mathcal{J}_1 \rangle = \mathcal{T}(X,Y)$. Therefore, $rank(\mathcal{T}(X,Y) : \mathcal{J}_1) = 0$. □

2.2 Relative Rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{S}(X,Y)$ and modulo $\mathcal{E}(X,Y)$

In this section, we describe the relative rank $\mathcal{T}(X,Y)$ modulo $\mathcal{S}(X,Y)$ and modulo $\mathcal{E}(X,Y)$. Morover, we also give examples of the relative rank $\mathcal{T}(X,Y)$ modulo $\mathcal{S}(X,Y)$ and modulo $\mathcal{E}(X,Y)$ to obtain that the relative rank are infinite for Y is a finite set.

Lemma 2.4. *[9] Let $B \subseteq \mathcal{T}(X,Y)$ such that any $\alpha \in B$ is not injective and its restriction to Y is either a bijection or not surjective. Moreover, let $A \subseteq \mathcal{T}(X,Y)$ be a generating set of $\mathcal{T}(X,Y)$ modulo B. Then there are $\mu, \nu \in A$ such that μ is injective and $\nu|_Y$ is a proper surjection.*

Theorem 2.5. *[9] Let $\mu \in \mathcal{T}(X,Y)$ be an injection such that $d(\mu|_Y) = |X|$ and let $\nu \in \mathcal{T}(X,Y)$ such that $\nu|_Y$ is surjective and $k(\nu|_Y) = |X|$. Then $\langle \mathcal{S}(X,Y), \mu, \nu \rangle = \mathcal{T}(X,Y)$; i.e. $rank(\mathcal{T}(X,Y) : \mathcal{S}(X,Y)) \leq 2$.*

Theorem 2.6. *[9] Let $\mu \in \mathcal{T}(X,Y)$ be an injection with $d(\mu|_Y) = |X|$ and let $\nu \in \mathcal{T}(X,Y)$ such that $\nu|_{im(\mu^2)} = (\mu^2)^{-1}\mu$. Then $\langle \mathcal{E}(X,Y), \mu, \nu \rangle = \mathcal{T}(X,Y)$; i.e. $rank(\mathcal{T}(X,Y) : \mathcal{E}(X,Y)) \leq 2$.*

Corollary 2.7. *[9] $rank(\mathcal{T}(X,Y) : \mathcal{S}(X,Y)) = 2$ and $rank(\mathcal{T}(X,Y) : \mathcal{E}(X,Y)) = 2$.*

Proof. By Lemma 2.4, we obtain that $rank(\mathcal{T}(X,Y) : \mathcal{S}(X,Y)) \geq 2$ and $rank(\mathcal{T}(X,Y) : \mathcal{E}(X,Y)) \geq 2$. By Proposition 2.5 and 2.6, we obtain that $rank(\mathcal{T}(X,Y) : \mathcal{S}(X,Y)) \leq 2$ and $rank(\mathcal{T}(X,Y) : \mathcal{E}(X,Y)) \leq 2$. Altogether, $rank(\mathcal{T}(X,Y) : \mathcal{S}(X,Y)) = 2$ and $rank(\mathcal{T}(X,Y) : \mathcal{E}(X,Y)) = 2$. □

Next, we give examples to show that the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{S}(X,Y)$ and modulo $\mathcal{E}(X,Y)$ are infinite when $2 \leq |Y| < \aleph_0$.

Example 2.8. Let X be an infinite set and let Y be a finite subset of X contains at least two elements. Then $rank(\mathcal{T}(X,Y) : \mathcal{S}(X,Y))$ is infinite.

Solution. Assume that there is a finite set $A \subseteq \mathcal{T}(X,Y) \setminus \mathcal{S}(X,Y)$ such that $\langle \mathcal{S}(X,Y), A \rangle = \mathcal{T}(X,Y)$. Let $Y := \{y_1, y_2, ..., y_n\}$ such that $|Y| = n$ and let $\{x_2, x_3, ..., x_n\} \subseteq X \setminus Y$ be an (n-1)-elements set. Define sets A' and C by as follows:

$$A' := \{\alpha \in A : \text{there is } \bar{x} \in \ker(\alpha) \text{ with } Y \subseteq \bar{x}\}$$

and

$$C := \{B \subseteq X \setminus Y : \text{there is } \alpha \in A' \text{ with } B \cup Y \in \ker(\alpha)\}.$$

Then there exists $K \in \mathcal{P}(X \setminus (Y \cup \{x_2, x_3, ..., x_n\})) \setminus C$. Let $\tilde{K} := X \setminus (K \cup Y \cup \{x_2, x_3, ..., x_n\})$. Therefore, we define a mapping β by

$$x\beta := \begin{cases} y_1 & ; \quad x \in K \cup Y \\ y_2 & ; \quad x \in \tilde{K} \cup \{x_2\} \\ y_k & ; \quad x = x_k \text{ and } 3 \leq k \leq n. \end{cases}$$

It is easy to verify that $\beta \in \mathcal{T}(X, Y)$ and there exists $\bar{x} \in \ker(\beta)$ with $Y \subseteq \bar{x}$ but $\beta \notin A'$ and $im(\beta) = Y$. Since $\beta \in \mathcal{T}(X, Y)$, there are $\gamma_1, \gamma_2, ..., \gamma_n \in \mathcal{S}(X, Y) \cup A$ such that

$$\beta = \gamma_1\gamma_2 \cdots \gamma_n.$$

Since $im(\beta) = Y$, we have $rank(\gamma_1) = |Y|$ and $\ker(\beta) = \ker(\gamma_1)$. Since $|Y| \geq 2$ and all elements of Y belong to one kernel block, we get that $\gamma_1 \notin \mathcal{S}(X, Y)$; i.e. $\gamma_1 \in A$. In particular, $\gamma_1 \in A'$. This gives $K \in C$ which is a contradiction. Therefore, $rank(\mathcal{T}(X, Y) : \mathcal{S}(X, Y))$ is infinite.

Example 2.9. Let $X = \mathbb{N}$ and $Y = \{1, 2\}$. Then $rank(\mathcal{T}(X, Y) : \mathcal{E}(X, Y))$ is infinite.

Solution. Assume that there is a finite set $A \subseteq \mathcal{T}(X, Y)$ such that $\langle \mathcal{E}(X, Y), A \rangle = \mathcal{T}(X, Y)$. For $\alpha \in A \setminus B$, where $B := \{\beta \in \mathcal{T}(X, Y) : rank(\beta) = 1\}$, there exists $n_\alpha \in \mathbb{N}$ with $(1, n_\alpha) \notin \ker(\alpha)$. Define $m := \max\{n_\alpha : \alpha \in A \setminus B\}$ and define a mapping $\alpha : X \to Y$ by

$$x\alpha := \begin{cases} 1 & ; \quad x \leq m + 1 \\ 2 & ; \quad x > m + 1. \end{cases}$$

Clearly, $\alpha \in \mathcal{T}(X, Y)$. Since $\alpha \in \mathcal{T}(X, Y)$, there are $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in \mathcal{E}(X, Y) \cup A$ such that $\alpha = \varepsilon_1\varepsilon_2 \cdots \varepsilon_n$. Since $rank(\alpha) = 2$, the image of ε_i is a transversal of $\ker(\varepsilon_{i+1})$, for $1 \leq i \leq n-1$. Therefore, $im(\varepsilon_1\varepsilon_2 \cdots \varepsilon_n) = im(\varepsilon_n) = im(\alpha)$ and $ker(\varepsilon_1\varepsilon_2 \cdots \varepsilon_n) = \ker(\varepsilon_1) = \ker(\alpha)$.

By definition of mapping α, we have $(1, k) \in \ker(\alpha)$ for $1 \leq k \leq m + 1$. In particular, if $(1, 2) \in \ker(\alpha)$, then $\varepsilon_1 \notin \mathcal{E}(X, Y)$. Thus $\varepsilon_1 \in A$; i.e. there exists n_{ε_1} with $n_{\varepsilon_1} < m + 1$ and $(1, n_{\varepsilon_1}) \notin \ker(\varepsilon_1) = \ker(\alpha)$, a contradiction.

Example 2.10. Let X be an infinite set and Y be a finite subset of X with at least two elements. Then $rank(\mathcal{T}(X, Y) : \mathcal{E}(X, Y))$ is infinite.

Solution. Suppose that $Y = \{x_1, x_2, x_3, ..., x_n\}$ is a finite subset of X and $|Y| = n$. Assume that there exists a finite set $A \subseteq \mathcal{T}(X,Y)$ with $\langle \mathcal{E}(X,Y), A\rangle = \mathcal{T}(X,Y)$. Let

$$B := \{\beta \in \mathcal{T}(X,Y) : rank(\beta) < n\}.$$

For $\alpha \in A \setminus B$, there is element $x_\alpha \in X$ with $(x_1, x_\alpha) \notin \ker(\alpha)$. Since X is an infinite set, there are $y_2, y_3, y_4, ..., y_{n-1} \in X \setminus (\{x_\alpha : \alpha \in A \setminus B\} \cup \{x_1, x_2\})$. Then we define a mapping $\alpha : X \to Y$ by

$$x\alpha := \begin{cases} x_1 & ; \quad x \in \{x_\alpha : \alpha \in A \setminus B\} \cup \{x_1, x_2\} \\ x_2 & ; \quad x = y_2 \\ \vdots & \qquad \vdots \\ x_{n-1} & ; \quad x = y_{n-1} \\ x_n & ; \quad otherwise. \end{cases}$$

Since we assume that $\langle \mathcal{E}(X,Y), A\rangle = \mathcal{T}(X,Y)$, we have $\alpha = \varepsilon_1\varepsilon_2\varepsilon_3...\varepsilon_m$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_m \in \mathcal{E}(X,Y) \cup A$. Therefore, $im(\varepsilon_i)$ is a transversal of $\ker(\varepsilon_{i+1})$ for all $1 \leq i \leq m-1$ because of $rank(\alpha) = n = |Y|$. Hence

$$\ker(\varepsilon_1\varepsilon_2\varepsilon_3...\varepsilon_m) = \ker(\varepsilon_1) = \ker(\alpha).$$

Thus, we can consider two cases:

Case 1. If $\varepsilon_1 = \beta \in A \setminus B$, then $(x_1, x_\beta) \notin \ker(\beta) = \ker(\varepsilon_1) = \ker(\alpha)$. That contradicts with the assumtion of mapping α because $(x_1, x_\beta) \in ker(\alpha)$.

Case 2. If $\varepsilon_1 \in \mathcal{E}(X,Y)$, then $(x_1, x_2) \notin \ker(\varepsilon_1) = \ker(\alpha)$ which contradicts with $(x_1, x_2) \in \ker(\alpha)$.

Hence, $rank(\mathcal{T}(X,Y) : \mathcal{E}(X,Y))$ is infinite.

3 Conclusions

In Subsection 2.1, we studied the properties of the defect and the infinite contraction index of a transformation in $\mathcal{T}(X,Y)$. We also determined the relative rank of $\mathcal{T}(X,Y)$ modulo the top $\mathcal{J}$ class. In Subsection 2.2, we described the relative rank $\mathcal{T}(X,Y)$ modulo $\mathcal{S}(X,Y)$ and modulo $\mathcal{E}(X,Y)$. In addition, we gave examples of the relative rank $\mathcal{T}(X,Y)$ and modulo $\mathcal{S}(X,Y)$ modulo $\mathcal{E}(X,Y)$, where Y is a finite set.

In future work, we can study other kinds of structures of transformation semigroup with restricted range.

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