International Journal of Mathematics and
Computer Science, **17**(2022), no. 3, 1193–1199

A Refinement of the KSOR Iterative Method

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(Received March 1, 2022, Accepted April 4, 2022)

Abstract

We introduce a refinement formulation of the KSOR method and establish the rate of convergence of the introduced formulas. We compare a new formula (RKSOR) with the classical form (KSOR). The efficient performance of the new form is established theoretically and confirmed through a numerical example. The decrease in the required number of iterations for convergence is established through the calculation of the spectral radius of the iteration matrices. The calculations and graphs are performed with the help of the computer algebra software Mathematica.

Key words and phrases: Linear systems, KSOR method, iterative and refinement technique.
AMS (MOS) Subject Classifications: 65F10, 15A06, 15A30.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

1 Introduction

The question of solving a large system of algebraic equations is a fundamental question in most modern modeling issues. Any linear system of algebraic equations can be written in matrix form as:

$$Ax = b, \tag{1.1}$$

where $A \in R^{nxn}$ is a coefficient nonsingular matrix, $b \in R^n$ is a known column of constants and x is an unknown vector. Theoretically, $x = A^{-1}b$ is known as the exact solution of the system (1.1). An efficient direct method for solving such systems requires about$(n^3/3)$ operations which is not suitable for large sparse systems. So an iterative method seems to be an appropriate choice especially when the convergence of the method up to the required accuracy is achieved within n steps because of the equivalence between the evaluation process in the iterative methods and matrix vector multiplications, in the worst case. Usually, coefficient matrices contain many zeros (sparse matrices). In iterative methods, unlike direct methods, zeros do not affect the computational work. It is well known that any splitting of the coefficient matrix A, $A = M - N$, with nonsingular matrix M, defines an iterative technique, [1, 2]

$$Mx^{[k+1]} = Nx^{[k]} + b,\ k = 0, 1, 2, ... \tag{1.2}$$

The spectral radius of the iteration matrix $\rho(M^{-1}N)$ is taken as the measure of convergence of the iterative technique, the method with smaller spectral radius of its iteration matrix is known as asymptotically faster. Also, the splitting, $A = D - L - U$, where D is the diagonal part of the matrix A, and $-L$, $-U$ are the strictly lower and upper triangular parts of A, respectively [2, 3] is used in the matrix reformulation of the standard stationary iterative techniques. We are interested in the KSOR method, [3–6]

$$x^{[k+1]} = M^{-1}Nx^{[k]} + M^{-1}\omega^* b = T_{KSOR}x^{[k]} + C_{KSOR}, \tag{1.3}$$

where

$$M = (1 + \omega^*)D - \omega^* L,\ N = D + \omega^* U \, and \, \omega^* \in R - [-2, 0]. \tag{1.4}$$

2 A Refinement of the KSOR (RKSOR) Method

Stationary iterative techniques are characterized by the fixed construction of their iteration matrices during the evaluation process. In general, in stationary iterative methods the iteration matrix is calculated only in the first step

and used in the consecutive iterations, so the computational overheads are of order at most n^2 per iteration. Refinement techniques are considered in many publications, [7–10]. For a convergent iterative method, the speed of convergence can be doubled with the refinement treatment when the same iterative technique is used. We consider the refinement of the KSOR method introduced by the second author in 2012, [5]. The achievement in the speed of convergence of the refinement treatments dominates the increase in computational costs appears in the first step. The basic idea in the refinement treatment is the use of a virtual step ($x^{[vir]}$) like the case of double sweep methods or the symmetric and unsymmetric techniques but without reversing the ordering of the equations.
The general iterative technique (1.2) can be written as

$$x^{[vir]} = M^{-1}Nx^{[k]} + M^{-1}b \tag{2.5}$$

and this virtual calculated data is used in a subsequent iteration as

$$x^{[k+1]} = M^{-1}Nx^{[vir]} + M^{-1}b,\ k = 0, 1, 2, ... \tag{2.6}$$

Which can be rearranged in the form

$$x^{[k+1]} = (M^{-1}N)^2x^{[k]} + (I + M^{-1}N)M^{-1}b,\ k = 0, 1, 2, ... \tag{2.7}$$

Although, it appears simple and straightforward treatment it includes some hidden consequences. The KSOR method employ the splitting

$$\omega^* A = M - N \tag{2.8}$$

As descibed in (1.3) and (1.4).
The refinement formula of the KSOR (RKSOR) method is

$$x^{[k+1]} = [M^{-1}N]^2x^{[k]} + [I + M^{-1}N]\omega^* M^{-1}b = T_{RKSOR}x^{[k]} + C_{RKSOR} \tag{2.9}$$

Remark 2.1. *From (2.9), (1.3) and (1.4), We find*

$$T_{RKSOR} = T_{KSOR}^2 \, and \, C_{RKSOR} = [I + T_{KSOR}]C_{KSOR} \tag{2.10}$$

Theorem 2.2. *Let $A \in R^{nxn}$ with $a_{ii} \neq 0$, Then $\rho(T_{RKSOR}) \geq 1/(1 + \omega^*)^2$ which implies that the RKSOR method can converge for $\omega^* \in R - [-2, 0]$.*

Proof.
The proof is straightforward and use the properties of determinant multiplications of square matrices like the original proof in, [5] and used in the same steps in [4].

Theorem 2.3. *Let A be a strictly diagonally dominant (SDD) matrix, then for every $\omega^* \in R - [-2, 0]$ the RKSOR method is convergent for any initial guess $x^{[0]}$; moreover, it is two times faster than the KSOR method.*

Proof.
Let x^* be the exact solution of linear system (1.1). The KSOR method is convergent [3], due to the strict diagonal dominance of the matrix A. If $x^{[k+1]}$ be the $(k+1)^{th}$ approximation to the solution of linear system (1.1) obtained by the RKSOR method (2.9), then we have

$$\begin{aligned} \|x^{[k+1]} - x^*\| &= \|x^{[vir]} + \omega^* M^{-1}(b - Ax^{[vir]}) - x^*\| \\ &\leq \|x^{[vir]} - x^*\| + |\omega^*|\|M^{-1}\|\|b - Ax^{[vir]}\| \end{aligned}$$

From the diagonal dominance, it is known that $||x^{[vir]} - x^*||$ converges to the zero vector and $||b - Ax^{[vir]}||$ converges to the zero vector. Accordingly, $||x^{[k+1]} - x^*||$ converges to the zero vector which means the convergence of the method. Hence, the RKSOR method converges to the solution of linear system (1.1). The RKSOR method is two times faster than the KSOR method is clear due to the definition of the rate of convergence in terms of the logarithm of the spectral radius of the iteration matrix, [1, 2].

3 Numerical Example

To illustrate the theoretical results, we consider a numerical example which appears in the discretization of Poisson's equation over square grid using natural ordering of the grid points, [2, 4, 7].

Example 3.1. We consider the linear system of equations, [7]

$$\begin{aligned} &4x_1 - x_2 - x_4 = 1, \\ &-x_1 + 4x_2 - x_3 - x_5 = 0, \\ &-x_2 + 4x_3 - x_6 = 0, \\ &-x_1 + 4x_4 - x_5 = 0, \\ &-x_2 - x_4 + 4x_5 - x_6 = 0, \\ &-x_3 - x_5 + 4x_6 = 0. \end{aligned} \tag{3.11}$$

The spectral radii of the iteration matrices of KSOR and RKSOR methods for the system (3.11) are $\rho(T_{KSOR}) = 0.113636$ and $\rho(T_{RKSOR}) = 0.0129132$. From tables (1, 2), the refinement approach duplicates the speed of convergence (theorem 2.3).

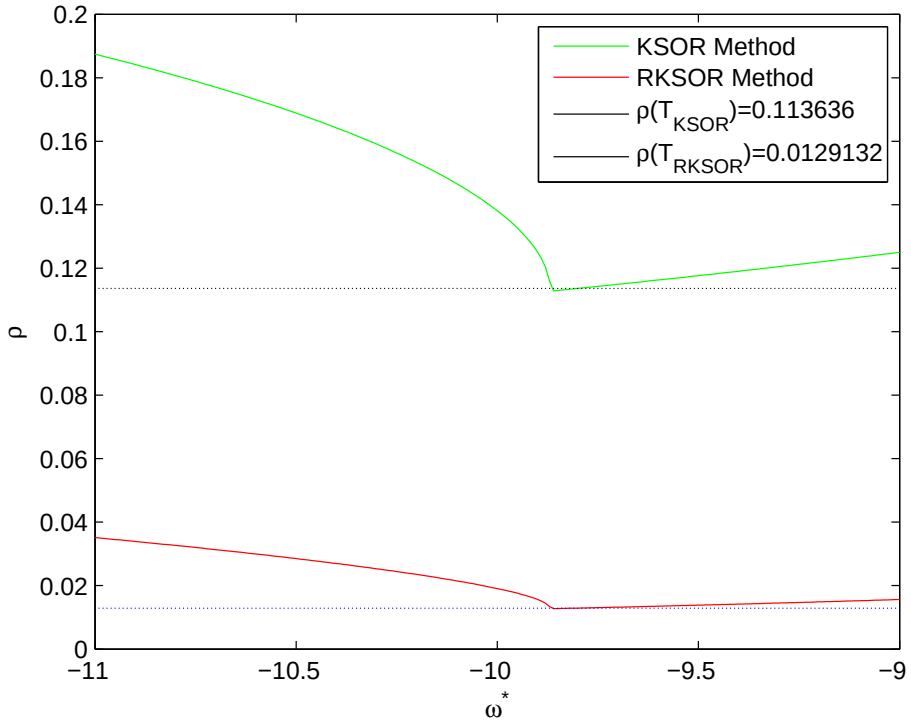

Figure 1: The behavior of the spectral radius for the KSOR and RKSOR methods

Table 1: The solution of the linear system (3.11) by the KSOR method at $\omega^* = -9.8$.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$	$x_5^{[k]}$	$x_6^{[k]}$
0	0	0	0	0	0	0
1	0.278409	0.0775116	0.0215799	0.0775116	0.0431599	0.0180242
2	0.289932	0.0899356	0.0276047	0.0839276	0.0485186	0.0191452
3	0.293867	0.0927888	0.0280266	0.0857862	0.0495336	0.0194179
4	0.294732	0.0931053	0.0281426	0.0860982	0.0496692	0.0194569
5	0.294809	0.0931608	0.0281558	0.0861219	0.0496867	0.0194610
6	0.294822	0.0931667	0.0281571	0.0861278	0.0496891	0.0194616
7	0.294824	0.0931676	0.0281573	0.0861283	0.0496894	0.0194617
8	0.294824	0.0931677	0.0281573	0.0861274	0.0496894	0.0194617

4 Disscusion and Results

It is generally accepted that iterative techniques for large sparse systems are preferable to direct methods [1, 2]. The slow convergence rates of some

Table 2: The solution of the linear system (3.11) by the RKSOR method at $\omega^* = -9.8$.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$	$x_5^{[k]}$	$x_6^{[k]}$
0	0	0	0	0	0	0
1	0.289932	0.0899356	0.0276047	0.0839276	0.0485186	0.0191452
2	0.294732	0.0931053	0.0281426	0.0860982	0.0496692	0.0194569
3	0.294822	0.0931667	0.0281571	0.0861278	0.0496891	0.0194616
4	0.294824	0.0931677	0.0281573	0.0861274	0.0496894	0.0194617

methods restricted the efficient use of iterative methods. There are different techniques introduced to increase the convergence of the stationary iterative technique, double sweep methods are one of the methods used to decrease the spectral radius of the iteration matrix. In double sweep methods the computational costs occur in the first step (calculation of the iteration matrix). In 2012 Youssef [5] introduced the KSOR method and in 2013 Youssef and Taha [3] introduced the MKSOR method. In 2019 Constantlnescu et al. [4] introduced the PKSOR method which gives slightly small number of iterations. In 2021 Ming-Ming Xu et al. [11] used half-sweep SOR iterative method. In this work, we introduced a refinement of the KSOR which reduces the number of iteration to the half. It is clear from (2.10) and tables (1, 2). Tables (1, 2) show that RKSOR method converges twice faster than KSOR method.

5 Conclusion

The main objective is to introduce a reformulation of the KSOR method. This formulation reduces the number of iterations and in the same time the increase in the computational costs is acceptable. The objective is highly achieved through using the refinement approach. As expected, the refinement of the KSOR method (RKSOR) is two times faster than the KSOR method which is clear from the relation between the iteration matrices, while the computational costs excluding the first step are the same.

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