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Local analytic solutions of an iterative functional differential equation near resonance

Jiraphorn Somsuwan Phanwan, Adchara Kumla, Wiratchada Kalaoka

Department of Mathematics
Faculty of Science
Nakhon Phanom University
Nakhon Phanom, Thailand

email: jira.somsu@hotmail.com, adchara_k@npu.ac.th,
wiratchadakalaoka@gmail.com

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Abstract

In this paper, we find an analytic solution of a functional differential equation of the form $x'(z) = x(az + \frac{b}{x'(z)})$ in the complex field. By reducing the equation to an auxiliary equation, we describe an existence theorem to the analytic solutions of the primary equation.

1 Introduction

Many authors have considered the functional differential equation of the form

$$x^{(n)}(z) = f(z, x^{(n_1)}(z - \tau_1(z)), x^{(n_2)}(z - \tau_2(z)), \dots, x^{(n_k)}(z - \tau_k(z))), \quad (1.1)$$

where all $n_i \geq 0, \tau_i \geq 0 :$ In 1984, Eder [3] studied solutions of the functional differential equation $x'(z) = x(x(z))$ by using the Banach fixed point theorem. Then Si and Cheng [5] presented the existence of analytic solutions of the functional differential equation

$$x'(z) = x(az + bx(z)),$$

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where a, b are nonzero complex constants.

Taking $f(z,x) = x$, $n = 1, n_1 = 0, \tau_1(z) = (1-a)z - \frac{b}{x'(z)}$, and $x^{(n_2)}(z - \tau_2(z)) = \ldots = x^{(n_k)}(z - \tau_k(z)) = 0$, in (1.1) , we get

$$x'(z) = x\left(az + \frac{b}{x'(z)}\right), \tag{1.2}$$

where a, b are nonzero complex constants and $x'(z) \neq 0$.
In this paper, we describe the existence of analytic solutions of (1.2).

To show an analytic solution of the equation (1.2), let $y(z) = az + \frac{b}{x'(z)}$. Then, for any complex number z_0, we obtain $x(z) = x(z_0) + b\int_{z_0}^{z} \frac{1}{y(s)-as}ds$. Thus $x(y(z)) = x(z_0) + b\int_{z_0}^{y(z)} \frac{1}{y(s)-as}ds$. Therefore,

$$\frac{b}{y(z) - az} = x(z_0) + b\int_{z_0}^{y(z)} \frac{1}{y(s) - as}ds. \tag{1.3}$$

If z_0 is a fixed point of $y(z)$, then

$$x(z_0) = \frac{b}{z_0 - az_0}. \tag{1.4}$$

Moreover, differentiating both sides of (1.3) with respect to z, yields

$$(ab - by'(z))((y(y(z)) - ay(z))) = by'(z)(y(z) - az)^2. \tag{1.5}$$

We reduce Eq.(1.5) with $y(z) = g(\gamma g^{-1}(z))$ to the auxiliary equation

$$(abg'(z) - b\gamma g'(\gamma z))(g(\gamma^2 z) - ag(\gamma z)) = b\gamma g'(\gamma z) \cdot (g(\gamma z) - ag(z))^2, \tag{1.6}$$

where $g(z)$ satisfies the initial value conditions $g(0) = \xi$ and $g'(0) = \beta \neq 0$.

2 Main results

To construct analytic solutions of (1.6), we separate our study on the conditions of the parameter γ as follows:

$(H1)$ $0 < |\gamma| < 1$;

$(H2)$ $\gamma = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a Brjuno number; That is, $B(\theta) = \sum_{k=0}^{\infty} \log q_{k+1}/q_k < \infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ;

$(H3)$ $\gamma = e^{2\pi iq/p}$ for some integers $p \in \mathbb{N}$ with $p \geq 2$ and $q \in \mathbb{Z} \setminus \{0\}$, and $\gamma \neq e^{2\pi il/k}$ for all $1 \leq k \leq p-1$ and $l \in \mathbb{Z} \setminus \{0\}$.

Theorem 2.1. *Let γ satisfy the condition $(H1)$. Then the equation* (1.6) *has an analytic solution $g(z) = \xi + \beta z + \sum_{n=2}^{\infty} c_n z^n$, where $\xi = \frac{a-\gamma}{\gamma(1-a)}, a \neq 1$ and β is a nonzero complex number.*

Proof.

Substituting $g(z)$ into (1.6) and comparing coefficients of $z^n (n = 0, 1, 2, \ldots)$, we get $c_0 = \frac{a-\gamma}{\gamma(1-a)} = \xi$, $c_1 = \beta \neq 0$ and for $n \geq 1$

$$\begin{aligned} &\frac{ab(n+1)}{\gamma} c_{n+1}(a-\gamma)(1-\gamma^n) \\ &= \sum_{k=0}^{n-1}\sum_{i=0}^{n-k} b(k+1)c_i c_{k+1} c_{n-k-i}\gamma^{k+1}(\gamma^i - a)(\gamma^{n-k-i} - a) \\ &\quad + \sum_{k=0}^{n-1} b(k+1)c_{k+1}c_{n-k}\gamma^{n-k}(\gamma^{k+1} - a)(\gamma^{n-k} - a). \end{aligned}$$

Hence the sequence $\{c_n\}_{n=2}^{\infty}$ is successively determined by the expression in a unique manner. This implies that (1.6) has a formal power series solution. Next, we show that the power series $g(z)$ converges in a neighborhood of the origin. Since $\lim_{n\to\infty} \frac{1}{1-\gamma^n} = 1$ for $0 < |\gamma| < 1$, there exists a positive constant M such that

$$|c_{n+1}| \leq M\left[\sum_{k=0}^{n-1}\sum_{i=0}^{n-k} |c_{k+1}||c_i||c_{n-k-i}| + \sum_{k=0}^{n-1} |c_{k+1}||c_{n-k}|\right].$$

Now define a positive power series $\sum_{n=0}^{\infty} D_n z^n$, where $\{D_n\}_{n=0}^{\infty}$ is determined by $D_0 = |\xi|, D_1 = |\beta|$ and for $n \geq 1$

$$D_{n+1} = M\left[\sum_{k=0}^{n-1}\sum_{i=0}^{n-k} D_{k+1}D_i D_{n-k-i} + \sum_{k=0}^{n-1} D_{k+1}D_{n-k}\right].$$

It follows that $|c_n| \leq D_n$ for $n \geq 0$. That is, $\sum_{n=0}^{\infty} D_n z^n$ is a majorant series of $\sum_{n=0}^{\infty} c_n z^n$. We now show that $\sum_{n=0}^{\infty} D_n z^n$ is analytic in a neighborhood of the origin. From $D(z) = \sum_{n=0}^{\infty} D_n z^n$, we see that

$$\begin{aligned} D^3(z) &= |\xi|D^2(z) + |\xi|^2|\beta|z + \left(|\xi|^2 D(z) - |\xi|^3 - |\xi|^2|\beta|z\right) + \frac{1}{M}\left(D(z) - |\xi| - |\beta|z\right) \\ &\quad - \left[D^2(z) - |\xi|D(z) - |\xi||\beta|z - |\xi|D(z) + |\xi|^2 + |\beta||\xi|z\right] \\ &= (|\xi| - 1)D^2(z) + (|\xi|^2 + \frac{1}{M} + 2|\xi|)D(z) - \frac{|\beta|}{M}z - |\xi|^3 - \frac{|\xi|}{M} - |\xi|^2. \end{aligned}$$

Consider the equation

$$T(z,D) = D^3 - (|\xi|-1)D^2 - (|\xi|^2 + \frac{1}{M} + 2|\xi|)D + \frac{|\beta|}{M}z + |\xi|^3 + \frac{|\xi|}{M} + |\xi|^2.$$

Since T is continuous in a neighborhood of the origin and $T(0,\xi) = 0$ and $T'_D(0,\xi) = -\frac{1}{M} \neq 0$, the implicit function theorem implies that there exists a unique function $D(z)$ which is analytic in a neighborhood of the origin with a positive radius. Since $D(z)$ is a majorant sereis of $g(z)$, $g(z)$ is also analytic in a neighborhood of the origin with a positive radius. This completes the proof.

Now, we consider an analytic solution $g(z)$ in the case of γ satisfying $(H2)$ under the Brjuno condition. We will recall the definition of Brjuno number and some basic facts. As stated in [4], for a real number θ, we let $[\theta]$ be the integer part of θ and $\{\theta\} = \theta - [\theta]$ be a fractional part of θ. Observe that if θ is an irrational number, then it has a unique expression of the Gauss's continued fraction $\theta = d_0 + \theta_0 = d_0 + \frac{1}{d_1+\theta_1} = \cdots$, denoted simply by $\theta = [d_0; d_1, \ldots, d_n, \ldots]$, where $d_j's$ and $\theta_j's$ are calculated by the following algorithm:

(a) $d_0 = [\theta], \theta_0 = \{\theta\}$;

(b) $d_n = \left[\frac{1}{\theta_{n-1}}\right], \theta_n = \left\{\frac{1}{\theta_{n-1}}\right\}$ for all $n \geq 1$.

Define the sequences $\{p_n\}_{n\in N}$ and $\{q_n\}_{n\in\mathbb{N}}$ by the following recursive relation:

$$\begin{aligned} &q_{-2} = 1, \qquad &q_{-1} = 0, \qquad &q_n = d_n q_{n-1} + q_{n-2};\\ &p_{-2} = 0, \qquad &p_{-1} = 1, \qquad &p_n = d_n p_{n-1} + p_{n-2}. \end{aligned}$$

Note that $\frac{p_n}{q_n} = [d_0; d_1, \ldots, d_n]$. For each $\theta \in \mathbb{R}\backslash\mathbb{Q}$, we consider an arithmetical function $B(\theta) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n}$. When θ satisfies condition $B(\theta) < \infty$, we call it a Brjuno number. Consider $\theta = [d_0; d_1, \ldots, d_n, \ldots]$, such that for each $n \geq 0$, $d_{n+1} \leq ce^{d_n}$, where c is a positive constant. We can show that θ is a Brjuno number but is not a Diophantine number. Therefore, Brjuno condition is weaker than the Diophantine condition. So, the case $(H2)$ contains both Diophantine condition and a part of γ near resonance.

Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$ and $\{q_n\}_{n\in\mathbb{N}}$ be the sequence of partial denominators of the Gauss's continued fraction for θ. As in [4], let

$$A_k = \left\{n \geq 0 : \| n\theta \| \leq \frac{1}{8q_k}\right\}, \qquad E_k = \max\left\{q_k, \frac{q_{k+1}}{4}\right\}, \qquad \beta_k = \frac{q_k}{E_k}.$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any nonnegative integer n, we define

$$l_k(n) = \max\left\{(1+\beta_k)\frac{n}{q_k} - 2, (m_n\beta_k + n)\frac{1}{q_k} - 1\right\},$$

where $m_n = \max\{j : 0 \leq j \leq n, j \in A_k^*\}$. Let $h_k : \mathbb{N} \to \mathbb{R}^+$ be a function defined by

$$h_k(n) = \begin{cases} \frac{m_n+\beta_k n}{q_k} - 1 & \text{if } m_n + q_k \in A_k^*, \\ l_k(n) & \text{if } m_n + q_k \notin A_k^*. \end{cases}$$

Let $g_k(n) := \max\{h_k(n), [n/q_k]\}$, and let $k(n)$ be defined by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Note that $k(n)$ is non-decreasing.

Lemma 2.2. (*Davie's lemma [2]*).
Let $K(n) = n \log 2 + \sum_{k=0}^{k(n)} g_k(n) \log(2q_{k+1})$. Then

(a) there is a universal constant $\xi > 0$ (independent of n and θ) such that $K(n) \leq n(B(\theta) + \xi)$,

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ for all n_1 and n_2,

(c) $-\log|\gamma^n - 1| \leq K(n) - K(n-1)$.

Theorem 2.3. *Assume that γ satisfies the condition $(H2)$. Then there exists an analytic solution $g(z) = \xi + \beta z + \sum_{n=2}^{\infty} c_n z^n$ where $\xi = \frac{a-\gamma}{\gamma(1-a)}, a \neq 1$ and β is a nonzero complex number.*

Proof.
We see that the sequence $\{c_n\}_{n=0}^{\infty}$ can be defined similar to the proof of Theorem 2.1. That is, $c_0 = \xi$ and $c_1 = \beta \neq 0$ and for $n \geq 1$

$$\begin{aligned} &\frac{ab(n+1)}{\gamma} c_{n+1}(a-\gamma)(1-\gamma^n) \\ &= \sum_{k=0}^{n-1}\sum_{i=0}^{n-k} b(k+1)c_i c_{k+1} c_{n-k-i}\gamma^{k+1}(\gamma^i - a)(\gamma^{n-k-i} - a) \\ &\quad + \sum_{k=0}^{n-1} b(k+1)c_{k+1}c_{n-k}\gamma^{n-k}(\gamma^{k+1} - a)(\gamma^{n-k} - a). \end{aligned}$$

Since $|\gamma| = 1$, there exists a positive constant N so that

$$|c_{n+1}| \leq \frac{N}{|1-\gamma^n|}\left[\sum_{k=0}^{n-1}\sum_{i=0}^{n-k} c_i c_{k+1} c_{n-k-i} + \sum_{k=0}^{n-1} c_{k+1} c_{n-k}\right]. \tag{2.7}$$

To construct a governing series of $g(z)$, we let $S(z) = \sum_{k=0}^{\infty} s_n z^n, s_0 = |\xi|, s_1 = |\beta|, s_{n+1} = N[\sum_{k=0}^{n-1}\sum_{i=0}^{n-k} |s_{k+1}||s_i||s_{n-k-i}| + \sum_{k=0}^{n-1} |s_{k+1}||s_{n-k}|]$. From this construction, we can show that a power series $S(z) = \sum_{n=0}^{\infty} s_n z^n$ satisfies the implicit functional equation

$$R(z,S) = S^3 - (|\xi|-1)S^2 - (|\xi|^2 + \frac{1}{M} + 2|\xi|)S + \frac{|\beta|}{M}z + |\xi|^3 + \frac{|\xi|}{M} + |\xi|^2$$

with $R(0,\xi) = 0$ and $R'_S(0,\xi) = \frac{-1}{M} \neq 0$. Then $S(z)$ converges in a neighborhood of the origin. Hence, there exists a positive constant T such that $s_n \leq T^n$ for $n \geq 0$.

Let K be a function defined in Lemma 2.2. By mathematical induction, we can show that for $n \in \mathbb{N} \cup \{0\}$ $|c_n| \leq s_n e^{k(n)}$. In fact, $|c_0| = |\xi| = s_0 e^{(0)} = s_0$. For an inductive proof, we assume $|c_i| \leq s_i e^{k(i-1)}, i \leq m$. From (2.7), we get

$$|c_{m+1}| \leq \frac{N}{|1-\gamma^m|}[\sum_{k=0}^{m-1}\sum_{i=0}^{m-k} s_{k+1}s_i s_{m-k-i} e^{k(k)} e^{k(i-1)} e^{k(m-k-i-1)} + \sum_{k=0}^{n-1} s_{k+1}s_{m-k} e^{k(k)} e^{k(m-k-1)}].$$

Lemma 2.2 yields $k(k) + k(i-1) + k(m-k-i-1) \leq \log|\gamma^m - 1| + k(m)$ and $k(k) + k(m-k-1) \leq \log|\gamma^m - 1| + k(m)$. Then

$$|c_{m+1}| \leq Ne^{k(m)}[\sum_{k=0}^{m-1}\sum_{i=0}^{m-k} s_{k+1}s_i s_{m-k-i} + \sum_{k=0}^{n-1} s_{k+1}s_{m-k}] = e^{k(m)}s_{m+1}$$

as desired. From Lemma 2.2 and $\{s_n\}$ bounded, we get $k(n) \leq (n-1)(B(\theta)+\epsilon)$, for some $\epsilon > 0$ and $|c_n| \leq T^n e^{(n-1)(B(\theta)+\epsilon)}$. Hence $\lim_{n\to\infty} |c_n|^{\frac{1}{n}} \leq Te^{B(\theta)+\epsilon}$. This implies that $g(z)$ has a convergence radius at least $(Te^{B(\theta)+\xi})^{-1}$.

Finally, we consider the case of γ satisfying $(H3)$. In this case, γ is not only on the unit circle, but also a root of unity. Let $\{E_n\}_{n=0}^{\infty}$ be a sequence defined by $E_0 = |\xi|, E_1 = |\beta|$ with $\Gamma = \max\left\{\frac{1}{|\gamma-1|}, \frac{1}{|\gamma^2-1|}, \ldots, \frac{1}{|\gamma^{p-1}-1|}\right\}$, and

$$E_{n+1} = \Gamma N[\sum_{k=0}^{n-1}\sum_{i=0}^{n-k} E_i E_{k+1} E_{n-k-i} + \sum_{k=0}^{n-1} E_{k+1}E_{n-k}], \tag{2.8}$$

where ξ, β are defined in Theorem 2.1 and N is a positive constant defined as in the proof of Theorem 2.3.

Theorem 2.4. *Assume that* γ *satisfies condition* $(H3)$. *Let* $g(z) = \sum_{n=0}^{\infty} c_n z^n$ *be a power series determined by* $c_0 = \xi = \frac{a-\gamma}{\gamma(1-a)}, a \neq 1$ *and* $c_1 = \beta$ *is a nonzero complex number and*

$$\frac{ab(n+1)}{\gamma}(a-\gamma)(1-\gamma^n)c_{n+1} = \Phi(n,\gamma), n = 1,2,3,\dots$$

$$\text{where } \Phi(n,\gamma) = \sum_{k=0}^{n-1}\sum_{i=0}^{n-k}\gamma^{k+1}(\gamma^i - a)(\gamma^{n-k-i} - a)c_i c_{k+1} c_{n-k-i} + \sum_{k=0}^{n-1}\gamma^{n-k}(\gamma^{n-k} - a)(\gamma^{k+1} - a)c_{k+1}c_{n-k}.$$

If $\Phi(vp,\gamma) \neq 0$ *for some* $v = 1,2,\dots,$ *then the equation* (1.6) *has no analytic solution in a neighborhood of the origin.*

Otherwise, if $\Phi(vp,\gamma) = 0$ *for* $v = 1,2,\dots,$ *then the equation* (1.6) *has an analytic solution* $g(z)$ *in a neighborhood of the origin such that* $g(0) = \xi, g'(0) = \beta \neq 0$ *and* $g^{(vp+1)}(0) = (vp+1)!c_{vp+1}$, *where* c_{vp+1} *is an arbitrary constant satisfying* $|c_{vp+1}| \leq E_{vp+1}$ *and* $\{E_n\}_{n=0}^{\infty}$ *is defined in* (2.8).

Proof.

If $\Phi(vp,\gamma) \neq 0$ for some positive number v, then $\frac{b(vp+1)}{\gamma}a(a-\gamma)(1-\gamma^{vp})c_{vp+1} \neq 0$. But $(H3)$ implies $1-\gamma^{vp} = 0$, which is a contradiction. This concludes that the equation (1.6) has no analytic solution in a neighborhood of the origin.

Assume that $\Phi(vp,\gamma) = 0$ for $v = 1,2,\dots.$ Then $\frac{b(vp+1)}{\gamma}a(a-\gamma)(1-\gamma^{vp})c_{vp+1} = 0$. So there are infinitely many choices of c_{vp+1}. It follows that $|c_{vp+1}| \leq E_{vp+1}$, where E_{vp+1} is defined in (2.8). Note that $|1-\gamma^n|^{-1} \leq \Gamma$, for $n \neq vp$. We can see that

$$|c_{n+1}| \leq \Gamma N[\sum_{k=0}^{n-1}\sum_{i=0}^{n-k}|c_{k+1}||c_i||c_{n-k-i}| + \sum_{k=0}^{n-1}|c_{k+1}||c_{n-k}|],$$

for $n \neq vp, v = 1,2,\dots.$ Likewise, the remaining proof is similar to Theorem 2.1. Consider the implicit functional equation $H(z,E) = E^3 - (|\xi|-1)E^2 - (|\xi|^2 + \frac{1}{M} + 2|\xi|)E + \frac{|\beta|}{M}z + |\xi|^3 + \frac{|\xi|}{M} + |\xi|^2$. Since $H(0,\xi) = 0, H'_E(0,\xi) = -\frac{1}{M} \neq 0$, the implicit function theorem implies that there exists a unique function $E(z)$ which is analytic in a neighborhood of the origin with a positive radius. Moreover, $|c_n| \leq E_n$ for $n \geq 0$. That is, $E(z)$ is a majorant series of $g(z)$. Then $g(z)$ converges in a neighborhood of the origin. This completes the proof.

Theorem 2.5. *Let $g(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic solution in a neighborhood of the origin of the equation* (1.6)*, with $g(0) = \xi, g'(0) = \beta \neq 0$, which is obtained from Theorem* 2.1*, Theorem* 2.3*, or Theorem* 2.4*. Then the equation* (1.5) *has an analytic solution of the form $y(z) = g(\gamma g^{-1}(z))$ in a neighborhood of the number ξ.*

Proof.
Since $g'(0) = \beta \neq 0, g^{-1}(z)$ is analytic in a neighborhood of $g(0) = \xi$.
From $y(z) = g(\gamma g^{-1}(z))$, we get $y'(z) = \frac{\gamma g'(\gamma g^{-1}(z))}{g'(g^{-1}(z))}$. Then

$$\begin{aligned}(ab - by'(z))(y(y(z)) - ay(z)) &= \frac{b\gamma g'(\gamma g^{-1}(z)) \cdot (g(\gamma g^{-1}(z) - ag(g^{-1}(z))}{g'(g^{-1}(z)} \\ &= \frac{b\gamma g'(\gamma g^{-1}(z))}{g'(g^{-1}(z))} \cdot (g(\gamma g^{-1}(z)) - az)^2 \\ &= by'(z)(y(z) - az)^2.\end{aligned}$$

That is, $y(z) = g(\gamma g^{-1}(z))$ is an analytic solution of (1.5).
Next, we construct an analytic solution of (1.2) from an analytic solution of (1.5). Assume that $x(z)$ is an analytic solution of the functional differential equation (1.2) in a neighborhood of the origin. Since $x(z)$ is analytic in a neighborhood of the origin, $x(z)$ can be represented in Taylor's series

$$x(z) = \sum_{n=0}^{\infty} \frac{x^{(n)}(\xi)}{n!} \cdot (z - \xi)^n = x(\xi) + x'(\xi)(z - \xi) + \frac{x''(\xi)(z - \xi)^2}{2!} + \cdots .$$

We need to determine the derivatives $x^{(n)}(\xi), n = 0, 1, 2, \ldots$. First of all, in view of (1.4) and $y(\xi) = \xi$, we have $x(\xi) = \frac{b}{\xi - a\xi}$. From (1.2), we get $x'(\xi) = \frac{b}{\xi - a\xi}$. Next, by calculating the derivatives of (1.2), we obtain $x''(\xi) = x'(a\xi + \frac{b}{x'(\xi)}) \cdot (a - \frac{bx''(\xi)}{(x'(\xi))^2})$. That is, $x''(\xi) = \frac{ab}{\mu - a\mu} \cdot \frac{1}{1+\mu - a\mu}$. Higher derivatives $x^{(m)}(z)$ at $z = \xi$ can be defined uniquely in similar manners. Hence, let $x^{(m)}(z) = \lambda_m$, the explicit form of our equation is

$$x(z) = \frac{b}{\xi - a\xi} + \frac{b}{\xi - a\xi}(z - \xi) + .\frac{ab}{\xi - a\xi} \cdot \frac{1}{(1 + \xi - a\xi)2!}(z - \xi)^2 + \cdots + \sum_{m=2}^{\infty} \frac{\lambda_m}{m!}(z - \xi)^m.$$

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