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Fractional Shehu Transform for Solving Fractional Differential Equations without Singular Kernel

Araya Wiwatwanich, Duangkamol Poltem

Department of Mathematics
Faculty of Science
Burapha University
Chonburi, Thailand

email: duangkamolp@buu.ac.th

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Abstract

In this paper, we derive the Shehu transform of the Caputo-Fabrizio fractional derivative without singular kernel. Moreover, we apply the proposed transform to solve Caputo-Fabrizio fractional differential equations and fractional integral equation. Furthermore, we present simulation examples to validate the result.

1 Introduction

Integral transforms are commonly used to convert a function to another in expectation to simplify any calculation on it. In handling with ordinary differential equations (ODEs), partial differential equations (PDEs), and fractional differential equations (FDEs), one of the most ordinary transforms is the Laplace transform. Nowadays, a number of variations of Laplace transform have been developed, such as, Shehu [1], Sumudu [2], Elzaki [3], Natural [4], Aboodh [5], Pourreza [6], Mohand [7], and Sawi [8].

The purpose of this work is to apply the Shehu transform to solve IVPs involving Caputo-Fabrizio fractional derivative [9]. This derivative operator

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has been discussed in several studies throughout this decade. The applications to well-known equations were investigated, such as, Fishers reaction-diffusion equation [10], Schrdinger equations [11], Korteweg-de Vries-Burgers Equation [12]. The role of Caputo-Fabrizio fractional derivative in medical field also gains higher interest. For example, it appears in the mathematical model of cancer chemotherapy effect [13] and in the analysis of HIV/AIDS epidemic [14].

In this paper, we propose the Shehu transform to solve Caputo-Fabrizio fractional differential equations and Caputo-Fabrizio fractional integral equation. In Section 2, we present some definitions, theorems and properties of both Shehu and Caputo-Fabrizio transform. In Section 3, we derive the formula for Shehu transform of Caputo-Fabrizio. In Section 4 and 5, numerical examples and conclusions are included, respectively.

2 Preliminaries and Notations

In this section, we present some basic ideas about the Shehu transform and Caputo-Fabrizio fractional derivative.

Definition 2.1. *The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions*

$$A = \{f(t) : \exists N, \eta_1, \eta_2 > 0, |f(t)| < N \exp\left(\frac{|t|}{\eta_i}\right), t \in (-1)^i \times [0, \infty)\}$$

by the following integral

$$\mathbb{S}[f(t)] = F(s,u) = \int_0^\infty \exp\left(\frac{-st}{u}\right) f(t)dt; \quad s > 0, u > 0. \tag{2.1}$$

where $\mathbb{S}[f(t)]$ is called the Shehu transform of time function (Variables s and u are the Shehu transform variables) which converges if the limit of the integral exists, and diverges otherwise.

Definition 2.2. *Let $F(u,s)$ be the Shehu transform of function $f(t)$. Then its inverse is given by*

$$\mathbb{S}^{-1}[F(u,s)] = f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{1}{u} \exp\left(\frac{st}{u}\right) F(u,s)ds. \tag{2.2}$$

The following useful formulas follow directly from equation (2.1)

1. $\mathbb{S}[1] = \frac{u}{s}$.

2. $\mathbb{S}[t] = \frac{u^2}{s^2}$.

3. $\mathbb{S}[\sin at] = \frac{au^2}{s^2 + a^2u^2}$.

4. $\mathbb{S}[\cos at] = \frac{us}{s^2 + a^2u^2}$.

5. $\mathbb{S}\left[\frac{t^n}{n!}\right] = \left(\frac{u}{s}\right)^{n+1}$, $n = 0, 1, 2, ...$

6. $\mathbb{S}[f(t) + g(t)] = \mathbb{S}[f(t)] + \mathbb{S}[g(t)]$. Linearity property.

Theorem 2.3. *Let $\mathbb{S}[f(t)] = F(s, u)$ and $\mathbb{S}[g(t)] = G(s, u)$. The Shehu transform of the function $(f * g)(t)$ is defined as*

$$\mathbb{S}[(f * g)(t)] = F(s, u)G(s, u). \tag{2.3}$$

Definition 2.4. *Let $f \in H^1(a, b), b > a$ and $n < \alpha \leq n + 1$. The Liouville-Caputo fractional derivative of order α is defined as*

$$^{C}D_{0,t}^{(\alpha)} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t f^{(n)}(\tau)(t - \tau)^{(n-\alpha-1)} d\tau, \ t > 0. \tag{2.4}$$

In 2015, Caputo and Fabrizio [9] proposed a modified fractional derivative without kernel singular which is applicable to many physical models.

Definition 2.5. *The fractional derivative of real order $0 < \alpha \leq 1$ for $f(t)$ in the Caputo-Fabrizio operator is defined as*

$$^{CF}D_{0,t}^{(\alpha)} f(t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t f'(\tau) \exp\left[-\alpha \frac{t - \tau}{1 - \alpha}\right] d\tau, \ t > 0 \tag{2.5}$$

where $M(\alpha)$ is the normalization function (any smooth positive function) such that $M(0) = M(1) = 1$, and $f \in H^1(a, b), b > a$. This derivative does not have singularities at $t = \tau$.

3 The Shehu Transform of Caputo-Fabrizio

Theorem 3.1. *If* $n-1 < \nu \leq n$, *n is positive integer,* $f(t)$, $f'(t), f''(t), ..., f^{(n)}(t)$, ${}^{CF}D_{0,t}^{(\nu)} f(t) \in A$ *then*

$$\mathbb{S}[{}^{CF}D_{0,t}^{\nu} f(t)] = \frac{s^n u^{-n+1} F(s,u) - \sum_{k=0}^{n-1} s^{n-k-1} u^{k-n+2} f^{(k)}(0)}{s + (\nu - n + 1)(u - s)}. \quad (3.6)$$

Proof. From Eq. (2.5), by setting $M(\alpha) = 1$ and taking the Shehu transform, we obtain

$$\mathbb{S}[{}^{CF}D_{0,t}^{(\alpha)} f(t)] = \frac{1}{1-\alpha} \int_0^{\infty} \exp\left[\frac{-st}{u}\right] \int_0^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau dt. \quad (3.7)$$

Applying the convolution theorem on right hand side of equation (3.7), we get

$$\begin{aligned} \mathbb{S}[{}^{CF}D_{0,t}^{(\alpha)} f(t)] &= \frac{1}{1-\alpha} \mathbb{S}[f'(t)] \mathbb{S}\left[\exp\left(\frac{-\alpha t}{1-\alpha}\right)\right], \\ &= \frac{1}{1-\alpha} \left(\frac{s}{u} F(s,u) - f(0)\right) \frac{u}{s + (\frac{\alpha}{1-\alpha})u}, \\ &= \frac{1}{1-\alpha} \left(\frac{s}{u} F(s,u) - f(0)\right) \frac{u(1-\alpha)}{s(1-\alpha) + \alpha u}, \\ &= \frac{sF(s,u) - uf(0)}{s + \alpha(u-s)}, \end{aligned}$$

where $\mathbb{S}[f(t)] = F(s,u)$.
Similarly,

$$\begin{aligned} \mathbb{S}[{}^{CF}D_{0,t}^{(\alpha+1)} f(t)] &= \frac{1}{1-\alpha} \mathbb{S}[f''(t)] \mathbb{S}\left[\exp\left(\frac{-\alpha t}{1-\alpha}\right)\right], \\ &= \frac{1}{1-\alpha} \left(\frac{s^2}{u^2} F(s,u) - \frac{s}{u} f(0) - f'(0)\right) \frac{u(1-\alpha)}{s(1-\alpha) + \alpha u}, \\ &= \frac{s^2 u^{-1} F(s,u) - sf(0) - uf'(0)}{s + \alpha(u-s)}. \end{aligned}$$

Finally,

$$\mathbb{S}[{}^{CF}D_{0,t}^{(\alpha+m)} f(t)] = \frac{s^{m+1} u^{-m} F(s,u) - \sum_{k=0}^{m} s^{m-k} u^{k-m+1} f^{(k)}(0)}{s + \alpha(u-s)}. \quad (3.8)$$

Let $\nu = \alpha + m$. We have $n - 1 < \nu \leq n$ and $\alpha = \nu - n + 1$, where n is a positive integer. Then the Shehu Transform of Caputo-Fabrizio is given by

$$\mathbb{S}[{}^{CF}D^{\nu}_{0,t}f(t)] = \frac{s^n u^{-n+1}F(s,u) - \sum_{k=0}^{n-1} s^{n-k-1}u^{k-n+2}f^{(k)}(0)}{s + (\nu - n + 1)(u - s)}. \tag{3.9}$$

□

4 Examples

Example 4.1. *Consider the following equation in the Caputo-Fabrizio sense:*

$${}^{CF}D^{\nu}_{0,t}(f(t)) = \lambda t, \tag{4.10}$$

with the initial conditions

$$f(0) = 0, \quad f'(0) = 1,$$

where $1 < \nu \leq 2$, $t > 0$ *and* λ *is any constant.*

The exact solution when $\nu = 2$ is given by

$$f(t) = \frac{\lambda t^3}{6} + t. \tag{4.11}$$

Therefore, applying the Shehu transform formula (3.9) yields

$$\frac{s^2 u^{-1}F(s,u) - sf(0) - uf'(0)}{s + (\nu - n + 1)(u - s)} = \frac{\lambda u^2}{s^2}, \tag{4.12}$$

where $\mathbb{S}[f(t)] = F(s,u)$. Substituting $f(0) = 0$, $f'(0) = 1$ and $n = 2$ into equation (4.12), we have

$$F(s,u) = \frac{\lambda u^3}{s^3}(2 - \nu) + \frac{\lambda \nu u^4}{s^4} - \frac{\lambda u^4}{s^4} + \frac{u^2}{s^2}. \tag{4.13}$$

Applying the inverse Shehu transfrom of equation (4.13), we obtain the following solution

$$f(t) = \lambda(2 - \nu)\frac{t^2}{2!} + \lambda(\nu - 1)\frac{t^3}{3!} + t. \tag{4.14}$$

In equation (4.14), we can directly observe that the solution is exact when $\nu = 2$. The graphical results of the solution for Eq. (4.10) is illustrated through Figures 1 for $\lambda = 1$ and various of ν. We observe that ν decreases resulting in the increases of the solution.

□

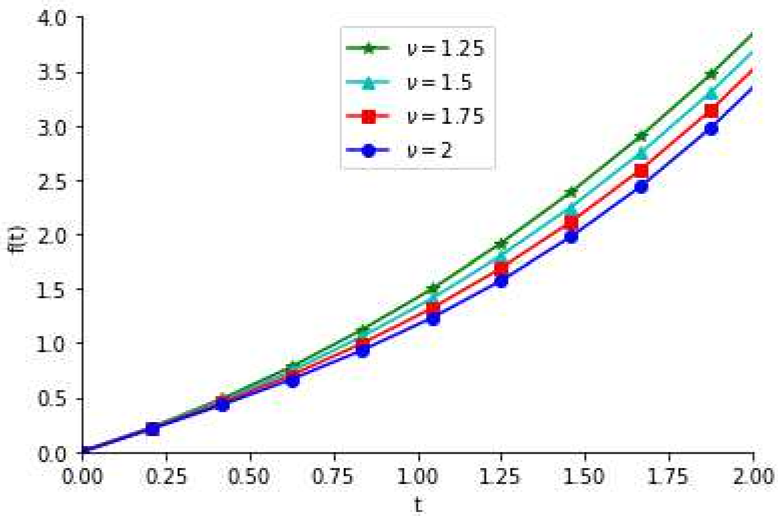

Figure 1: Solution of Eq. (4.10) for $\lambda = 1$ and various of ν.

Example 4.2. *Consider the following equation in the Caputo-Fabrizio sense:*

$$^{CF}D^{\nu}_{0,t}(f(t)) = f(t), \quad t > 0, \tag{4.15}$$

subject to the initial conditions

$$f(0) = 0, \quad f'(0) = 1,$$

where $1 < \nu \leq 2$ *and* $t > 0$.

Applying the Shehu transform on both sides of equation (4.15), we get

$$\frac{s^2u^{-1}F(s,u) - sf(0) - uf'(0)}{s + (\nu - n + 1)(u - s)} = F(s,u).$$

Substituting the given initial conditions and simplifying, we have

$$F(s,u) = \frac{u^2}{(s-u)[s-(1-\nu)u]}. \tag{4.16}$$

Simplifying the equation (4.16), we obtain the following

$$F(s,u) = \frac{1}{\nu}\left[\frac{u}{s-u} - \frac{u}{s-(1-\nu)u}\right]. \tag{4.17}$$

Therefore, we can applied the inverse Shehu transform, which yields the solution

$$f(t) = \frac{1}{\nu}(e^t - e^{(1-\nu)t}). \tag{4.18}$$

In equation (4.18), we can directly observe that the solution is exact when $\nu = 1$. The graphical results of the solution for Eq. (4.15) is presented as shown in Figures 2 for various of ν. We observe that ν decreases resulting in the increases of the solution.

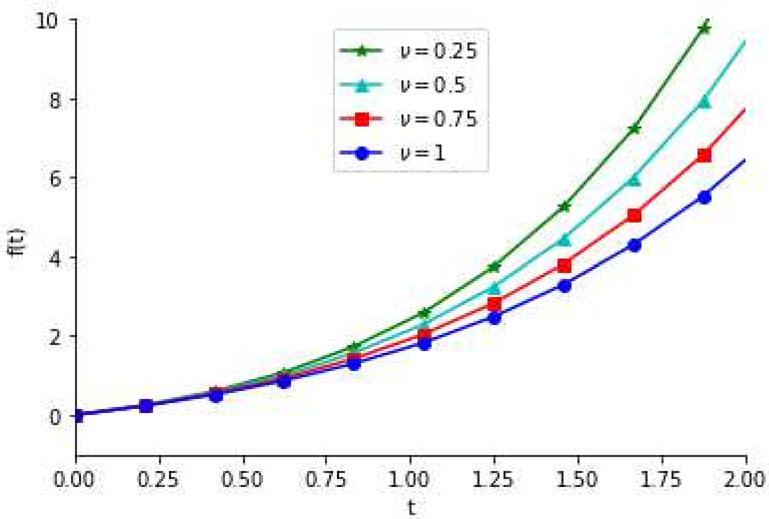

Figure 2: Solution of Eq. (4.18) for various of ν.

□

Example 4.3. *Consider the following integral equation in the Caputo-Fabrizio sense:*

$$^{CF}D^{\nu}_{0,t}(f(t)) = 1 - \int_0^x f(t)dt, \quad t > 0, \tag{4.19}$$

subject to the initial conditions

$$f(0) = 0,$$

where $0 < \nu \leq 1$ *and* $t > 0$.

Applying the Shehu transform on both sides of equation (4.15), we obtain

$$\frac{sF(s,u) - uf(0)}{s + \nu - (u - s)} = \frac{u}{s} - \frac{u}{s}F(s,u).$$

Substituting the given initial conditions and simplifying, we have

$$F(s,u) = \frac{us}{s^2+u^2} + \frac{\nu u^2}{s^2+u^2} + \frac{\nu us}{s^2+u^2}. \tag{4.20}$$

Therefore, we can apply the inverse Shehu transform which yields the solution

$$f(t) = \cos t + \nu \sin t - \nu \cos t. \tag{4.21}$$

In equation (4.21), we can directly obtain the exact solution when $\nu = 1$ which is

$$f(t) = \sin t.$$

Figures 3 shows the graphical results of the solution for Eq. (4.19) for various of ν.

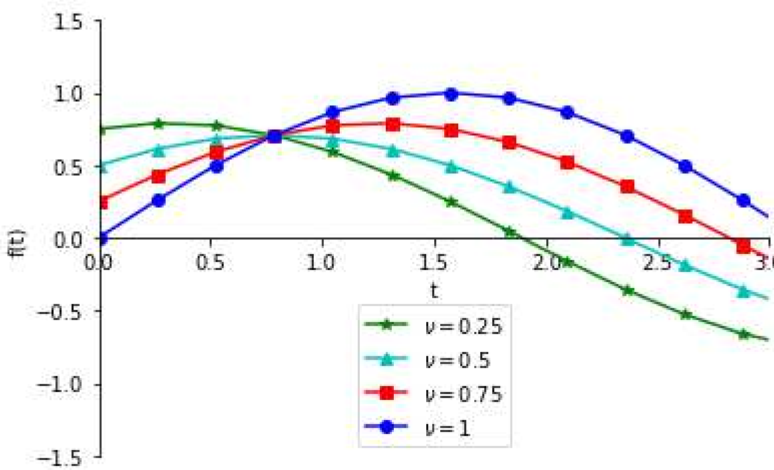

Figure 3: Solution of Eq. (4.21) for various of ν.

□

5 Conclusions

In this paper, the Shehu transform for solving fractional differential equations and fractional integral equations without a singular kernel is presented. Illustrative examples demonstrate the applicability of the results. In future work, another method like the variation method homotopy method will be applied to solve nonlinear fractions differential equations.

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