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Green's relations on ordered groupoids in terms of fuzzy semibipolar soft sets

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Abstract

The specification of Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered semigroups has been used to study the decomposition of ordered semigroups. Several researchers pointed out that such relations can be described in terms of fuzzy set theory. In this paper, we indicate that soft set theory, which is an extended concept of fuzzy set theory, can do the same. Based on the interesting point, we introduce the concept of fuzzy semibipolar soft sets and provide a corresponding example. At this point, some related properties are investigated. Related to the aforementioned accomplishments, the concept of fuzzy semibipolar soft right ideals, fuzzy semibipolar soft left ideals, and fuzzy semibipolar soft ideals is proposed in ordered groupoids. Then, Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered groupoids are characterized in terms of fuzzy semibipolar soft sets.

1 Introduction and earlier work

Green's relations play an important role in considering the structure of ordered semigroups. In particular, the decomposition of ordered semigroups is classified by the idea of Green's relations. The notion of Green's relations

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$\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered semigroups was proposed by Kehayopulu [1] in 1991. Then, this concept was developed in the condition of fuzzy subsets. That is, Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ were characterized in terms of fuzzy subsets of ordered groupoids by Kehayopulu [2] in 2007. This induced theorems in [3]. In 2012, Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ were characterized in terms of fuzzy subsets of ordered gamma-semigroups by Iampan and Siripitukdet [4]. Besides, the characterization of Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ in terms of fuzzy subsets of ordered gamma-groupoids was introduced by Iampan and Siripitukdet [5] in 2013. As reviewed above, we see that Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ can be described using fuzzy set theory. Furthermore, we observe that the notion of fuzzy sets emphasizes the study of the structure of algebraic systems. As is widely known, the concept of fuzzy sets is one of the useful mathematical tools introduced by Zadeh [6] in 1965. This theory is a classical tool for dealing with several uncertainty problems in informational and mathematical systems. In extended fuzziness works, the concept of soft set theory was introduced by Molodtsov [7] in 1999 who provided a general idea to define basic operations on soft sets. The motivation for the concept of soft sets comes from the need for a parameterization tool and then, the role of parameters becomes important for decision-making problems. Besides, the logical statemenst of soft set theory exist in many algebraic systems such as group, ring, semigroup, and so on. In the context of bipolarity, Shabir and Naz [8] proposed the notion of bipolar soft sets in 2013. This notion is generated by the hybridization of the bipolarity concept of Dubois and Prade [9] and soft set theory. They have studied the notion of the bipolarity of information in terms of soft sets. Bipolar soft set theory is described by two soft sets, one of which provides positive information and the other provides negative information. In 2014, Naz and Shabir [10] proposed the concept of fuzzy bipolar soft sets. At this point, this notion is defined for decision making problems in applications. In the structure of such a theory, the summation of the positive membership grade and the degree of negativity is less than or equal to 1. Bipolar-valued fuzzy membership grades are induced by distinct two parameters. Then, the same parameter-based fuzzy membership grades are attractive in this problem.

To the growth of ordered groupoid theory and soft set theory, we shall characterize Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered groupoids in terms of fuzzy bipolar soft set theory. In the main results of this paper, our contributions are as follows:

- To study a bipolar-valued fuzzy membership grade via the identical parameter, we introduce the concept of fuzzy semibipolar soft sets induced by fuzzy bipolar soft set theory in sense of Naz and Shabir [10]. A corresponding example is presented. Related properties are verified.

- We introduce the concept of fuzzy semibipolar soft right ideals, fuzzy semibipolar soft left ideals, and fuzzy semibipolar soft ideals in ordered groupoids. In the main point of this section, Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered groupoids are characterized in terms of fuzzy semibipolar soft sets.

In the following, we recall necessary definitions and properties which will be used in the subsequent section. Throughout this paper, $\varDelta$ denotes a non-empty universal set. A groupoid is a set $\varDelta$ together with a binary operation $*$ on $\varDelta$, denoted by $(\varDelta, *)$. Generally, if $(\varDelta, *)$ is a groupoid, then $\acute{\xi} * \grave{\xi}$ is denoted by $\acute{\xi}\grave{\xi}$ for all $\acute{\xi}, \grave{\xi} \in \varDelta$. Given two non-empty subsets $\acute{\varXi}$ and $\grave{\varXi}$ of a groupoid $(\varDelta, *)$, the product $\acute{\varXi} * \grave{\varXi}$ (simply $\acute{\varXi}\grave{\varXi}$) is defined by

$$\acute{\varXi}\grave{\varXi} = \{\acute{\xi}\grave{\xi} : \acute{\xi} \in \acute{\varXi} \text{ and } \grave{\xi} \in \grave{\varXi}\}.$$

Let $\leq_\varDelta$ be a given binary relation on $\varDelta$. An ordered groupoid, denoted by $(\varDelta, *, \leq_\varDelta)$, is a groupoid $(\varDelta, *)$ whose elements of $\varDelta$ are partially ordered by $\leq_\varDelta$ satisfying the property that for all $\xi, \acute{\xi}, \grave{\xi} \in \varDelta$,

$$\acute{\xi} \leq_\varDelta \grave{\xi} \text{ implies } \acute{\xi}\xi \leq_\varDelta \grave{\xi}\xi \; and \; \xi\acute{\xi} \leq_\varDelta \xi\grave{\xi}.$$

For simplicity, We usually write $\varDelta$ instead of $(\varDelta, *, \leq_\varDelta)$. Recall that an ordered semigroup is defined as an ordered associative groupoid. To use the characterization of Green's relations, we shall review several notations and notions in an ordered groupoid $\varDelta$. A non-empty subset $\varXi$ of $\varDelta$ is called a right ideal (resp., a left ideal) of $\varDelta$ if

$$\varXi\varDelta \subseteq \varXi \text{ (resp., } \varDelta\varXi \subseteq \varXi)$$

and for all $\acute{\xi}, \grave{\xi} \in \varDelta$,

$$\acute{\xi} \in \varXi \text{ and } \varDelta \ni \grave{\xi} \leq_\varDelta \acute{\xi} \text{ imply } \grave{\xi} \in \varXi.$$

At this point, $\varXi$ is called an ideal of $\varDelta$ if $\varXi$ is both a right ideal and a left ideal of $\varDelta$ [3]. For each $\xi \in \varDelta$, $R(\xi)$ (resp., $L(\xi)$ and $I(\xi)$) is denoted as a right ideal (resp., a left ideal and an ideal) of $\varDelta$ generated by ξ. For a non-empty subset $\varXi$ of $\varDelta$, the notation $(\varXi]$ is a subset of $\varDelta$ defined by

$$(\varXi] := \{\acute{\xi} \in \varDelta : \acute{\xi} \leq_\varDelta \grave{\xi} \text{ for some } \grave{\xi} \in \varXi\}[3].$$

In what follows, for each $\xi \in \Delta$, it is true that

$$R(\xi) = (\xi \cup \xi\Delta], L(\xi) = (\xi \cup \Delta\xi] \text{ and } I(\xi) = (\xi \cup \xi\Delta \cup \Delta\xi \cup \Delta\xi\Delta]$$

[3]. The Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on Δ in [1] are defined as follows:

$$\mathcal{R} := \{(\acute{\xi}, \grave{\xi}) \in \Delta \times \Delta : R(\acute{\xi}) = R(\grave{\xi})\},$$

$$\mathcal{L} := \{(\acute{\xi}, \grave{\xi}) \in \Delta \times \Delta : L(\acute{\xi}) = L(\grave{\xi})\},$$

$$\mathcal{I} := \{(\acute{\xi}, \grave{\xi}) \in \Delta \times \Delta : I(\acute{\xi}) = I(\grave{\xi})\}.$$

Throughout this paper, U denotes a non-empty universal set. ι is said to be a fuzzy subset of U if it is a function from U to the closed unit interval $[0, 1]$ [6]. Throughout this paper, $\mathcal{F}(U)$ denotes a collection of all fuzzy subsets of U. In this way, 1_U denotes a fuzzy subset of U defined by $1_U(u) = 1$ for all $u \in U$, and 0_U denotes a fuzzy subset of U defined by $0_U(u) = 0$ for all $u \in U$ [6]. Obviously, 1_U is the greatest element of $\mathcal{F}(U)$ and 0_U is the least element of $\mathcal{F}(U)$. In the following, for $\iota, \kappa \in \mathcal{F}(U)$, the function $\iota + \kappa : U \to [0, 1]$ is defined by $(\iota + \kappa)(u) = \iota(u) + \kappa(u)$; i.e., a usual addition of $\iota(u)$ and $\kappa(u)$ for all $u \in U$, and $\iota \prec \kappa$ is used to denote $\iota(u) \leq \kappa(u)$ for all $u \in U$ [6]. Let $\{\iota_i : i \in I\}$ be a non-empty collection of all fuzzy subsets of U. Define

$$\widetilde{\bigwedge}_{i\in I}\iota_i : U \to [0, 1]|u \mapsto (\widetilde{\bigwedge}_{i\in I}\iota_i)(u) := \inf\{\iota_i(u) : i \in I\}$$

and

$$\widetilde{\bigvee}_{i\in I}\iota_i : U \to [0, 1]|u \mapsto (\widetilde{\bigvee}_{i\in I}\iota_i)(u) := \sup\{\iota_i(u) : i \in I\}.$$

Then $\widetilde{\bigwedge}_{i\in I}\iota_i, \widetilde{\bigvee}_{i\in I}\iota_i \in \mathcal{F}(U)$ [2]. Moreover, it is true that

$$\widetilde{\bigwedge}_{i\in I}\iota_i = \inf\{\iota_i : i \in I\} \text{ and } \widetilde{\bigvee}_{i\in I}\iota_i = \sup\{\iota_i : i \in I\}[2].$$

In the following, $\mathcal{P}(U)$ denotes a collection of all subsets of U. Let Ξ be a non-empty subset of Δ. If F is a function from Ξ to $\mathcal{P}(U)$, then (F, Ξ) is said to be a soft set over U with respect to Ξ. From the idea of the soft set, U is said to be a universe of all alternative objects of (F, Ξ), and Δ is said to be a set of all parameters of (F, Ξ), where parameters are attributes, characteristics or statements of alternative objects in U [7]. For any element $\xi \in \Xi$, $F(\xi)$ is considered as a set of ξ-approximate elements (or ξ-alternative objects) of (F, Ξ). For $\Xi := \{\xi_i : i \in \mathbb{N}\} \subseteq \Delta$, the NOT set of Ξ is denoted by $\sim \Xi$, where $\sim \Xi := \{\sim \xi_i := \text{not } \xi_i : \xi_i \in \Xi \text{ for } i \in \mathbb{N}\}$. Recall that $(F, \neg F, \varXi)$ is said to be a bipolar soft set over U if $F : \varXi \to \mathcal{P}(U)$ and $\neg F :\sim \varXi \to \mathcal{P}(U)$ are two functions such that $F(\xi)$ and $\neg F(\sim \xi)$ are disjoint for all $\xi \in \varXi$ [8]. $(F, \neg F, \varXi)$ is called a fuzzy bipolar soft set over U if $F : \varXi \to \mathcal{F}(U)$ and $\neg F :\sim \varXi \to \mathcal{F}(U)$ are functions such that $F(\xi) + \neg F(\sim \xi) \leq 1_U$ for all $\xi \in \varXi$ [10].

2 Main results

In this section, we introduce the concept of fuzzy semibipolar soft sets. Such an idea is induced by the concept of fuzzy bipolar soft sets in [10]. This notion is contained in Subsection 2.1. In Subsection 2.2, Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered groupoids are characterized in terms of fuzzy semibipolar soft sets.

2.1 Fuzzy semibipolar soft sets

In the subsection, we study the fundamentals of fuzzy semibipolar soft sets. In the following, $\varXi, \acute{\varXi}$ and $\grave{\varXi}$ denote three non-empty subsets of $\varDelta$.

Definition 2.1. *We say that $(F, \neg F, \varXi)$ is a fuzzy semibipolar soft set over U with respect to $\varXi$ if $F : \varXi \to \mathcal{F}(U)$ and $\neg F : \varXi \to \mathcal{F}(U)$ are disjoint functions such that $F(\xi) + \neg F(\xi) = 1_U$ for all $\xi \in \varXi$. We can represent a fuzzy semibipolar soft set $(F, \neg F, \varXi)$ over U as following form:*

$$(F, \neg F, \varXi) := \{(\xi, F(\xi), \neg F(\xi)) : F(\xi) + \neg F(\xi) = 1_U \textit{ for } \xi \in \varXi\}.$$

In what follows, we observe that the notion of Definition 2.1 is considered as a specific case of the concept of fuzzy bipolar soft sets. The interestedness of such a concept is a bipolar-valued fuzzy membership grade of one parameter exists obviously. That is, positive membership grades of $F(\xi)$ and negative membership grades of $\neg F(\xi)$ can be compared in the quantification based on the parameter element ξ. In view of Definition 2.1, we give the following example.

Example 2.2. *Let $U = \{u_i :=$ Alternative $i : i = 1, 2, 3, ...\}$ be a given universal set of arbitrary alternative elements. Let a subset $\varXi = \{\xi_i :=$ Parameter $i : i = 1, 2, 3, ...\}$ of $\varDelta$ be a set of arbitrary parameters. Define a function $F : \varXi \to \mathcal{F}(U)$ by*

$$F(\xi) = 1_U \ (\textit{resp.}, F(\xi) = 0_U)$$

for all $\xi \in \Xi$. *Define a function* $\neg F : \Xi \to \mathcal{F}(U)$ *by*

$$\neg F(\xi) = 0_U \ (resp., F(\xi) = 1_U),$$

for all $\xi \in \Xi$. *Then* $F(\xi) + \neg F(\xi) = 1_U + 0_U(\ resp.,0_U + 1_U) = 1_U$ *for all* $\xi \in \Xi$. *This means that* $(F, \neg F, \Xi)$ *is a fuzzy semibipolar soft set over* U. *In this case, we call* $(F, \neg F, \Xi)$ *the trivial fuzzy semibipolar soft set over* U.

Definition 2.3. *Let* $(F, \neg F, \acute{\Xi})$ *and* $(G, \neg G, \grave{\Xi})$ *be two fuzzy semibipolar soft sets over* U.

(*i*) *We say that* $(F, \neg F, \acute{\Xi})$ *is a fuzzy semibipolar soft subset of* $(G, \neg G, \grave{\Xi})$, *denoted by* $(F, \neg F, \acute{\Xi}) \Subset (G, \neg G, \grave{\Xi})$, *if it satisfies*

- $\acute{\Xi} \subseteq \grave{\Xi}$;
- $F(\xi) \prec G(\xi)$ *and* $\neg F(\xi) \succ \neg G(\xi)$ *for all* $\xi \in \acute{\Xi}$.

In the same way, we say that the pair $(G, \neg G, \grave{\Xi})$ *is a fuzzy semibipolar soft superset of* $(F, \neg F, \acute{\Xi})$.

(*ii*) *We say that* $(F, \neg F, \acute{\Xi})$ *is equal to* $(G, \neg G, \grave{\Xi})$ *if it satisfies*

- $(F, \neg F, \acute{\Xi}) \Subset (G, \neg G, \grave{\Xi})$;
- $(F, \neg F, \acute{\Xi}) \Supset (G, \neg G, \grave{\Xi})$.

Definition 2.4. *A relative null fuzzy semibipolar soft set over* U *is denoted by* $(\emptyset_\Xi, \neg\emptyset_\Xi, \Xi)$, *which is a fuzzy semibipolar soft set defined by*

$$\emptyset_\Xi(\xi) = 0_U \text{ and } \neg\emptyset_\Xi(\xi) = 1_U,$$

for all $\xi \in \Xi$. *A relative whole fuzzy semibipolar soft set over* U *is denoted by* $(U_\Xi, \neg U_\Xi, \Xi)$, *which is a fuzzy semibipolar soft set defined by*

$$U_\Xi(\xi) = 1_U \text{ and } \neg U_\Xi(\xi) = 0_U$$

for all $\xi \in \Xi$.

Using Definition 2.4, the following proposition follows easily.

Proposition 2.5. *If* $(F, \neg F, \Xi)$ *is a fuzzy semibipolar soft set over* U, *then*

(*i*) $(F, \neg F, \Xi) \Supset (\emptyset_\Xi, \neg\emptyset_\Xi, \Xi)$;

(*ii*) $(F, \neg F, \Xi) \Subset (U_\Xi, \neg U_\Xi, \Xi)$.

To construct a fuzzy semibipolar soft set for supporting the characterization concepts, we define two set-valued functions as follows.

Definition 2.6. *Let $\{(F_i, \neg F_i, \Xi) : i \in I\}$ be a non-empty collection of all fuzzy semibipolar soft sets over U with respect to Ξ. We define*

$$\widetilde{\bigcap}_{i\in I} F_i : \Xi \to \mathcal{F}(U) | \xi \mapsto (\widetilde{\bigcap}_{i\in I} F_i)(\xi) := \widetilde{\bigwedge}_{i\in I} F_i(\xi)$$

and

$$\widetilde{\bigcup}_{i\in I} \neg F_i : \Xi \to \mathcal{F}(U) | \xi \mapsto (\widetilde{\bigcup}_{i\in I} \neg F_i)(\xi) := \widetilde{\bigvee}_{i\in I} \neg F_i(\xi).$$

Proposition 2.7. *Let $\{(F_i, \neg F_i, \Xi) : i \in I\}$ be a non-empty collection of all fuzzy semibipolar soft sets over U with respect to Ξ. Then $(\widetilde{\bigcap}_{i\in I} F_i, \widetilde{\bigcup}_{i\in I} \neg F_i, \Xi)$ belongs to the collection of all fuzzy semibipolar soft sets over U.*

Proof. We prove that $\widetilde{\bigcap}_{i\in I} F_i$ and $\widetilde{\bigcup}_{i\in I} \neg F_i$ are well-defined. In fact, let $\xi \in \Xi$ be given. Then $\{F_i(\xi) : i \in I\}$ and $\{\neg F_i(\xi) : i \in I\}$ are non-empty set of all fuzzy subsets of U. As reviewed above, there exist $\widetilde{\bigwedge}_{i\in I} F_i(\xi)$ and $\widetilde{\bigvee}_{i\in I} \neg F_i(\xi)$ in $\mathcal{F}(U)$. If $\acute{\xi}, \grave{\xi} \in \Xi$ such that $\acute{\xi} = \grave{\xi}$, then it is clear that

$$(\widetilde{\bigcap}_{i\in I} F_i)(\acute{\xi}) = (\widetilde{\bigcap}_{i\in I} F_i)(\grave{\xi}) \text{ and } (\widetilde{\bigcup}_{i\in I} \neg F_i)(\acute{\xi}) = (\widetilde{\bigcup}_{i\in I} \neg F_i)(\grave{\xi}).$$

Since $F_i(\xi) + \neg F_i(\xi) = 1_U$ for all $i \in I$, we have

$$\inf_U \{F_i(\xi) : i \in I\} + \sup_U \{\neg F_i(\xi) : i \in I\} = 1.$$

Therefore, $(\widetilde{\bigcap}_{i\in I} F_i, \widetilde{\bigcup}_{i\in I} \neg F_i, \Xi)$ is a fuzzy semibipolar soft set over U.

Remark 2.8. *By Proposition 2.7, it is easy to see that*

$$(\widetilde{\bigcap}_{i\in I} F_i, \widetilde{\bigcup}_{i\in I} \neg F_i, \Xi) \Subset (F_j, \neg F_j, \Xi),$$

for every $j \in I$.

Proposition 2.9. *Let $\{(F_i, \neg F_i, \Xi) : i \in I\}$ be a non-empty collection of all fuzzy semibipolar soft sets over U with respect to Ξ. Then*

$$(\widetilde{\bigcap}_{i\in I} F_i, \widetilde{\bigcup}_{i\in I} \neg F_i, \Xi) = (\inf_\Xi \{F_i : i \in I\}, \sup_\Xi \{\neg F_i : i \in I\}, \Xi).$$

Proof. According to Proposition 2.7, we get $(\widetilde{\bigcap}_{i\in I}F_i, \widetilde{\bigcup}_{i\in I}\neg F_i, \Xi)$ is a fuzzy semibipolar soft set over U. By Remark 2.8, we have

$$(\widetilde{\bigcap}_{i\in I}F_i, \widetilde{\bigcup}_{i\in I}\neg F_i, \Xi) \Subset (F_j, \neg F_j, \Xi),$$

for every $j \in I$. Now, let $(G, \neg G, \Xi)$ be a fuzzy semibipolar soft set over U and $(G, \neg G, \Xi) \Subset (F_i, \neg F_i, \Xi)$ for all $i \in I$. Then

$$G(\xi) \prec F_i(\xi) \text{ and } \neg G(\xi) \succ \neg F_i(\xi),$$

for all $\xi \in \Xi$ and $i \in I$. From the proof of Proposition 2.7, it follows that

$$G(\xi) \prec \inf\{F_i(\xi) : i \in I\} =: (\widetilde{\bigcap}_{i\in I}F_i)(\xi)$$

and

$$\neg G(\xi) \succ \sup\{\neg F_i(\xi) : i \in I\} =: (\widetilde{\bigcup}_{i\in I}\neg F_i)(\xi)$$

for all $\xi \in \Xi$. This implies that $(G, \neg G, \Xi) \Subset (\widetilde{\bigcap}_{i\in I}F_i, \widetilde{\bigcup}_{i\in I}\neg F_i, \Xi)$.

Definition 2.10. *Let $\Lambda \subseteq \Xi$ be given. A characteristic function F_Λ concerning Λ, and a dual characteristic function $\neg F_\Lambda$ concerning Λ of F_Λ are set-valued functions of a fuzzy semibipolar soft set $(F_\Lambda, \neg F_\Lambda, \Xi)$ over U with respect to Ξ defined by*

$$F_\Lambda(\xi) = \begin{cases} 1_U & \text{if } \xi \in \Lambda, \\ 0_U & \text{if } \xi \notin \Lambda \end{cases}$$

and

$$\neg F_\Lambda(\xi) = \begin{cases} 0_U & \text{if } \xi \in \Lambda, \\ 1_U & \text{if } \xi \notin \Lambda, \end{cases}$$

for all $\xi \in \Xi$, respectively. Based on this point, we call $(F_\Lambda, \neg F_\Lambda, \Xi)$ the fuzzy semibipolar soft set over U concerning Λ. If $(F, \neg F, \Xi)$ is a fuzzy semibipolar soft set over U and $\xi \in \Xi$ such that

$$F(\xi) = 1_U \text{ and } \neg F(\xi) = 0_U,$$

then we say that $(F, \neg F, \Xi)$ contains ξ.

Remark 2.11. *By Definition 2.10, we see that for two subsets Θ and Λ of Ξ, $\Theta \subseteq \Lambda$ if and only if $(F_\Theta, \neg F_\Theta, \Xi) \Subset (F_\Lambda, \neg F_\Lambda, \Xi)$.*

Furthermore, an equality case is also true.

2.2 Green's relations on ordered groupoids in terms of fuzzy semibipolar soft sets

In this subsection, we use the basic notion of the previous subsection to study the characterization of Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered groupoids in terms of fuzzy semibipolar soft sets.

Definition 2.12. *Let $(\Xi, *, \leq_\Xi)$ be an ordered groupoid and $(F, \neg F, \Xi)$ a fuzzy semibipolar soft set over U with respect to Ξ.*

(i) $(F, \neg F, \Xi)$ is called a fuzzy semibipolar soft right ideal if it satisfies

- *$F(\acute{\xi}\grave{\xi}) \succ F(\acute{\xi})$ and $\neg F(\acute{\xi}\grave{\xi}) \prec \neg F(\acute{\xi})$ for all $\acute{\xi}, \grave{\xi} \in \Xi$.*
- *For any $\acute{\xi}, \grave{\xi} \in \Xi$, $\acute{\xi} \leq_\Xi \grave{\xi}$ implies $F(\acute{\xi}) \succ F(\grave{\xi})$ and $\neg F(\acute{\xi}) \prec \neg F(\grave{\xi})$.*

(ii) $(F, \neg F, \Xi)$ is called a fuzzy semibipolar soft left ideal if it satisfies

- *$F(\acute{\xi}\grave{\xi}) \succ F(\grave{\xi})$ and $\neg F(\acute{\xi}\grave{\xi}) \prec \neg F(\grave{\xi})$ for all $\acute{\xi}, \grave{\xi} \in \Xi$.*
- *For any $\acute{\xi}, \grave{\xi} \in \Xi$, $\acute{\xi} \leq_\Xi \grave{\xi}$ implies $F(\acute{\xi}) \succ F(\grave{\xi})$ and $\neg F(\acute{\xi}) \prec \neg F(\grave{\xi})$.*

(iii) $(F, \neg F, \Xi)$ is called a fuzzy semibipolar soft ideal if it is both a fuzzy semibipolar soft right ideal and a fuzzy semibipolar soft left ideal.

Example 2.13. *Let $\Delta = \{\xi_i : i \in \mathbb{N}\}$. Let $\Xi = \{\xi_1, \xi_2, \xi_3, \xi_4\} \subseteq \Delta$ be a set whose elements of Ξ are partially ordered by $\leq_\Xi$ satisfying*

$$\leq_\Xi := \{(\xi_1, \xi_1), (\xi_2, \xi_2), (\xi_3, \xi_3), (\xi_4, \xi_4), (\xi_2, \xi_3), (\xi_2, \xi_4)\}.$$

Define a binary operation $*$ on Ξ by multiplication rules as in Table 1.

Table 1: The table of multiplication rules on Ξ

$*$	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	ξ_1	ξ_1	ξ_1	ξ_1
ξ_2	ξ_1	ξ_2	ξ_2	ξ_2
ξ_3	ξ_1	ξ_2	ξ_3	ξ_4
ξ_4	ξ_1	ξ_2	ξ_4	ξ_3

It is a routine to verify that $(\Xi, *, \leq_\Xi)$ *is an ordered groupoid. Define a function* $F : \Xi \to \mathcal{F}(\Delta)$ *by*

$$F(\acute{\xi}) = 1_\Delta \text{ and } F(\grave{\xi}) = 0_\Delta$$

for all $\acute{\xi} \in \{\xi_1, \xi_2\}, \grave{\xi} \in \{\xi_3, \xi_4\}$. *Define a function* $\neg F : \Xi \to \mathcal{F}(\Delta)$ *by*

$$\neg F(\acute{\xi}) = 0_\Delta \text{ and } \neg F(\grave{\xi}) = 1_\Delta,$$

for all $\acute{\xi} \in \{\xi_1, \xi_2\}, \grave{\xi} \in \{\xi_3, \xi_4\}$. *Then* $F(\xi) + \neg F(\xi) = 1_\Delta$ *for all* $\xi \in \Xi$. *It follows that* $(F, \neg F, \Xi)$ *is a fuzzy semibipolar soft set over* Δ. *We verify that*

$$F(\acute{\xi}\grave{\xi}) \succ F(\acute{\xi}), \neg F(\acute{\xi}\grave{\xi}) \prec \neg F(\acute{\xi}), F(\acute{\xi}\grave{\xi}) \succ F(\grave{\xi}), \text{ and } \neg F(\acute{\xi}\grave{\xi}) \prec \neg F(\grave{\xi}),$$

for all $\acute{\xi}, \grave{\xi} \in \Xi$. *Furthermore, for any* $\acute{\xi}, \grave{\xi} \in \Xi$, $\acute{\xi} \leq_\Xi \grave{\xi}$ *implies* $F(\acute{\xi}) \succ F(\grave{\xi})$ *and* $\neg F(\acute{\xi}) \prec \neg F(\grave{\xi})$. *Thus* $(F, \neg F, \Xi)$ *is a fuzzy semibipolar soft right ideal and a fuzzy semibipolar soft left ideal. This implies that* $(F, \neg F, \Xi)$ *is a fuzzy semibipolar soft ideal.*

Proposition 2.14. *Let* $(\Xi, *, \leq_\Xi)$ *be an ordered groupoid and* $\{(F_i, \neg F_i, \Xi) : i \in I\}$ *a non-empty collection of all fuzzy semibipolar soft sets over* U *with respect to* Ξ. *If* $(F_i, \neg F_i, \Xi)$ *is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) for all* $i \in I$, *then* $(\widetilde{\bigcap}_{i\in I} F_i, \widetilde{\bigcup}_{i\in I} \neg F_i, \Xi)$ *is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal).*

Proof. Suppose that $(F_i, \neg F_i, \Xi)$ is a fuzzy semibipolar soft right ideal for all $i \in I$. By Proposition 2.7, it is true that $(\widetilde{\bigcap}_{i\in I} F_i, \widetilde{\bigcup}_{i\in I} \neg F_i, \Xi)$ is a fuzzy semibipolar soft set over U. Let $\acute{\xi}, \grave{\xi} \in \Xi$. Then

$$F_i(\acute{\xi}\grave{\xi}) \succ F_i(\acute{\xi}) \text{ and } \neg F_i(\acute{\xi}\grave{\xi}) \prec \neg F_i(\acute{\xi})$$

for all $i \in I$. Thus

$$(\widetilde{\bigcap}_{i\in I} F_i)(\acute{\xi}\grave{\xi}) := \inf\{F_i(\acute{\xi}\grave{\xi}) : i \in I\} \succ \inf\{F_i(\acute{\xi}) : i \in I\} =: (\widetilde{\bigcap}_{i\in I} F_i)(\acute{\xi})$$

and

$$(\widetilde{\bigcup}_{i\in I} \neg F_i)(\acute{\xi}\grave{\xi}) := \sup\{\neg F_i(\acute{\xi}\grave{\xi}) : i \in I\} \prec \sup\{\neg F_i(\acute{\xi}) : i \in I\} =: (\widetilde{\bigcup}_{i\in I} \neg F_i)(\acute{\xi}).$$

Assume that $\acute{\xi} \leq_\Xi \grave{\xi}$. Then

$$F_i(\acute{\xi}) \succ F_i(\grave{\xi}) \text{ and } \neg F_i(\acute{\xi}) \prec \neg F_i(\grave{\xi})$$

for all $i \in I$. Thus

$$(\widetilde{\bigcap}_{i\in I}F_i)(\acute{\xi}) := \inf\{F_i(\acute{\xi}) : i \in I\} \succ \inf\{F_i(\grave{\xi}) : i \in I\} =: (\widetilde{\bigcap}_{i\in I}F_i)(\grave{\xi})$$

and

$$(\widetilde{\bigcup}_{i\in I}\neg F_i)(\acute{\xi}) := \sup\{\neg F_i(\acute{\xi}) : i \in I\} \prec \sup\{\neg F_i(\grave{\xi}) : i \in I\} =: (\widetilde{\bigcup}_{i\in I}F_i)(\grave{\xi}).$$

Therefore, $(\widetilde{\bigcap}_{i\in I}F_i, \widetilde{\bigcup}_{i\in I}\neg F_i, \Xi)$ is a fuzzy semibipolar soft right ideal. The other arguments are similar.

Notation 2.15. *For an ordered groupoid $(\Xi, *, \leq_\Xi)$ and a fuzzy semibipolar soft set $\mathfrak{F} := (F, \neg F, \Xi)$ over U, we denote*

$$R_{\mathfrak{F}} := \left\{ \begin{array}{ll} \mathfrak{G} := (G, \neg G, \Xi) : & \mathfrak{G} \text{ is a fuzzy semibipolar soft} \\ & \text{right ideal over } U \text{ and } \mathfrak{F} \Subset \mathfrak{G} \end{array} \right\},$$

$$L_{\mathfrak{F}} := \left\{ \begin{array}{ll} \mathfrak{G} := (G, \neg G, \Xi) : & \mathfrak{G} \text{ is a fuzzy semibipolar soft} \\ & \text{left ideal over } U \text{ and } \mathfrak{F} \Subset \mathfrak{G} \end{array} \right\},$$

$$I_{\mathfrak{F}} := \left\{ \begin{array}{ll} \mathfrak{G} := (G, \neg G, \Xi) : & \mathfrak{G} \text{ is a fuzzy semibipolar soft} \\ & \text{ideal over } U \text{ and } \mathfrak{F} \Subset \mathfrak{G} \end{array} \right\}.$$

Remark 2.16. *By Notation 2.15, we observe that $(U_\Xi, \neg U_\Xi, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$). Then $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is a non-empty subset of a non-empty collection of all fuzzy semibipolar soft sets over U. By Proposition 2.7, we see that a fuzzy semibipolar soft set $(\widetilde{\bigcap}G, \widetilde{\bigcup}\neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) exists in such non-empty collection of all fuzzy semibipolar soft sets over U. Using Proposition 2.14, we obtain that the fuzzy semibipolar soft set $(\widetilde{\bigcap}G, \widetilde{\bigcup}\neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal).*

Proposition 2.17. *Let $(\Xi, *, \leq_\Xi)$ be a given ordered groupoid and let $\mathfrak{F} := (F, \neg F, \Xi)$ be a fuzzy semibipolar soft set over U. Then, a fuzzy semibipolar soft set $(\widetilde{\bigcap}G, \widetilde{\bigcup}\neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is an element of $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$).*

Proof. By Remark 2.16, we have a fuzzy semibipolar soft set $(\widetilde{\bigcap}G, \widetilde{\bigcup}\neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to $R_{\mathfrak{F}}$ is a fuzzy semibipolar soft right ideal. Suppose $(G, \neg G, \Xi) \in R_{\mathfrak{F}}$. Then $\mathfrak{F} \Subset (G, \neg G, \Xi)$. Let $\xi \in \Xi$. Then $F(\xi) \prec G(\xi)$ and $\neg F(\xi) \succ \neg G(\xi)$. Hence

$$F(\xi) \prec \inf\{G(\xi)\} =: (\widetilde{\bigcap} G)(\xi)$$

and

$$\neg F(\xi) \succ \sup\{\neg G(\xi)\} =: (\widetilde{\bigcup} \neg G)(\xi).$$

It follows that $\mathfrak{F} \Subset (\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \Xi)$. This means that $(\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \Xi) \in R_{\mathfrak{F}}$. The other arguments are similar.

Proposition 2.18. *Let $(\Xi, *, \leq_{\Xi})$ be a given ordered groupoid and let $\mathfrak{F} := (F, \neg F, \Xi)$ be a fuzzy semibipolar soft set over U. Then, a fuzzy semibipolar soft set $(\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is a fuzzy semibipolar soft subset of $(H, \neg H, \Xi)$ for every $(H, \neg H, \Xi) \in R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$).*

Proof. From Remark 2.8, the statement is true.

Definition 2.19. *Let $(\Xi, *, \leq_{\Xi})$ be an ordered groupoid and $\mathfrak{F} := (F, \neg F, \Xi)$ a fuzzy semibipolar soft set over U. A fuzzy semibipolar soft set $(G, \neg G, \Xi)$ over U is called a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) generated by $\mathfrak{F}$ if $(G, \neg G, \Xi) \in R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) and $(G, \neg G, \Xi) \Subset (H, \neg H, \Xi)$ for all $(H, \neg H, \Xi) \in R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$).*

Remark 2.20. *According to Propositions 2.17 and 2.18, it is easy to see that a fuzzy semibipolar soft set $(\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) generated by $\mathfrak{F}$.*

Proposition 2.21. *Let $(\Xi, *, \leq_{\Xi})$ be an ordered groupoid and $\mathfrak{F} := (F, \neg F, \Xi)$ a fuzzy semibipolar soft set over U. If $(G, \neg G, \Xi)$ is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) over U generated by $\mathfrak{F}$, then a fuzzy semibipolar soft set $(\widetilde{\bigcap} H, \widetilde{\bigcup} \neg H, \Xi)$ over U in which $(H, \neg H, \Xi)$ belongs to $R_{\mathfrak{F}}$(resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is equal to $(G, \neg G, \Xi)$.*

Proof. We only prove the case of a fuzzy semibipolar soft right ideal with the other arguments being similar. Suppose that $(G, \neg G, \Xi)$ is a fuzzy semibipolar soft right ideal over U generated by $\mathfrak{F}$. Then $(G, \neg G, \varXi) \in R_{\mathfrak{F}}$. By Remark 2.8, we get that a fuzzy semibipolar soft set $(\widetilde{\bigcap} H, \widetilde{\bigcup} \neg H, \varXi)$ over U in which $(H, \neg H, \varXi)$ belongs to $R_{\mathfrak{F}}$ is a fuzzy semibipolar soft subset of $(G, \neg G, \varXi)$. From Proposition 2.17, it follows that the fuzzy semibipolar soft set $(\widetilde{\bigcap} H, \widetilde{\bigcup} \neg H, \varXi)$ over U in which $(H, \neg H, \varXi)$ belongs to $R_{\mathfrak{F}}$ is an element of $R_{\mathfrak{F}}$. By the assumption, we obtain that the fuzzy semibipolar soft set $(\widetilde{\bigcap} H, \widetilde{\bigcup} \neg H, \varXi)$ over U in which $(H, \neg H, \varXi)$ belongs to $R_{\mathfrak{F}}$ is a fuzzy semibipolar soft superset of $(G, \neg G, \varXi)$. The proof is complete.

Notation 2.22. *For an ordered groupoid $(\varXi, *, \leq_{\varXi})$ and a fuzzy semibipolar soft set $\mathfrak{F} := (F, \neg F, \varXi)$ over U, we denote by $R(\mathfrak{F})$ (resp., $L(\mathfrak{F})$ and $I(\mathfrak{F})$) a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) over U generated by $\mathfrak{F}$. That is, a fuzzy semibipolar soft set $(\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \varXi)$ over U in which $(G, \neg G, \varXi)$ belongs to $R_{\mathfrak{F}}$ (resp., $L_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$) is denoted as $R(\mathfrak{F})$ (resp., $L(\mathfrak{F})$ and $I(\mathfrak{F})$).*

Notation 2.23. *For an ordered groupoid $(\varXi, *, \leq_{\varXi})$ and $\xi \in \varXi$, we denote*

$$R_{\xi} := \left\{ \begin{matrix} \mathfrak{F} := (F, \neg F, \varXi) : & \mathfrak{F} \text{ fuzzy semibipolar soft right ideal over } U, \\ & F(\xi) = 1_U \text{ and } \neg F(\xi) = 0_U \end{matrix} \right\},$$

$$L_{\xi} := \left\{ \begin{matrix} \mathfrak{F} := (F, \neg F, \varXi) : & \mathfrak{F} \text{ fuzzy semibipolar soft left ideal over } U, \\ & F(\xi) = 1_U \text{ and } \neg F(\xi) = 0_U \end{matrix} \right\},$$

$$I_{\xi} := \left\{ \begin{matrix} \mathfrak{F} := (F, \neg F, \varXi) : & \mathfrak{F} \text{ fuzzy semibipolar soft ideal over } U, \\ & F(\xi) = 1_U \text{ and } \neg F(\xi) = 0_U \end{matrix} \right\}.$$

Remark 2.24. *By Notation 2.23, we observe that $(U_{\varXi}, \neg U_{\varXi}, \varXi)$ belongs to R_{ξ} (resp., L_{ξ} and I_{ξ}). Then R_{ξ} (resp., L_{ξ} and I_{ξ}) is a non-empty subset of a non-empty collection of all fuzzy semibipolar soft sets over U. By Proposition 2.7, we see that a fuzzy semibipolar soft set $(\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \varXi)$ over U in which $(G, \neg G, \varXi)$ belongs to R_{ξ} (resp., L_{ξ} and I_{ξ}) exists in the non-empty collection of all fuzzy semibipolar soft sets over U. Using Proposition 2.14, we obtain that the fuzzy semibipolar soft set $(\widetilde{\bigcap} G, \widetilde{\bigcup} \neg G, \varXi)$ over U in which $(G, \neg G, \varXi)$ belongs to R_{ξ} (resp., L_{ξ} and I_{ξ}) is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal).*

Theorem 2.25. *Let $(\varXi, *, \leq_{\varXi})$ be an ordered groupoid and $\xi \in \varXi$. Then*

(*i*) $R_{\xi} = R_{(F_{\{\xi\}}, \neg F_{\{\xi\}}, \varXi)}$.

(*ii*) $L_{\xi} = L_{(F_{\{\xi\}}, \neg F_{\{\xi\}}, \varXi)}$.

(iii) $I_{\xi} = I_{(F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)}$.

Proof. In this theorem, we shall check (i). The remaining is similar, so we omit it.

Let $\mathfrak{F} := (F,\neg F,\Xi) \in R_{\xi}$. Then $\mathfrak{F}$ is a fuzzy semibipolar soft right ideal over U. Moreover, $F(\xi) = 1_U$ and $\neg F(\xi) = 0_U$. Thus $\mathfrak{F} \in R_{(F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)}$. Indeed, let $\acute{\xi} \in \Xi$.

Case 1. Assume $\xi = \acute{\xi}$. Then

$$F(\acute{\xi}) = F(\xi) = 1_U \text{ and } \neg F(\acute{\xi}) = \neg F(\xi) = 0_U.$$

Furthermore, observe that $F_{\{\xi\}}(\acute{\xi}) = 1_U$ and $\neg F_{\{\xi\}}(\acute{\xi}) = 0_U$. It follows that $F_{\{\xi\}}(\acute{\xi}) \prec F(\acute{\xi})$ and $\neg F_{\{\xi\}}(\acute{\xi}) \succ \neg F(\acute{\xi})$. Thus $\mathfrak{F} \Supset (F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)$.

Case 2. Assume $\xi \neq \acute{\xi}$. Then

$$F_{\{\xi\}}(\acute{\xi}) = 0_U \prec F(\acute{\xi}) \text{ and } \neg F_{\{\xi\}}(\acute{\xi}) = 1_U \succ F(\acute{\xi}).$$

Hence $\mathfrak{F} \Supset (F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)$.

Conversely, let $\mathfrak{G} := (G,\neg G,\Xi) \in R_{(F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)}$. Then $\mathfrak{G}$ is a fuzzy semibipolar soft right ideal over U and $\mathfrak{G} \Supset (F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)$. Hence

$$1_U = F_{\{\xi\}}(\xi) \prec G(\xi) \text{ and } 0_U = \neg F_{\{\xi\}}(\xi) \succ \neg G(\xi).$$

Observe that $G(\xi) = 1_U$ and $\neg G(\xi) = 0_U$. Therefore $\mathfrak{G} \in R_{\xi}$. This implies that $R_{\xi} = R_{(F_{\{\xi\}},\neg F_{\{\xi\}},\Xi)}$.

Proposition 2.26. *Let $(\Xi,*,\leq_{\Xi})$ be an ordered groupoid and let $\mathfrak{F} := (F,\neg F,\Xi)$ be a fuzzy semibipolar soft set over U. If $\mathfrak{F}$ is a fuzzy semibipolar soft right ideal, a fuzzy semibipolar soft left ideal, and a fuzzy semibipolar soft ideal, then $F^{-1}(1_U) = \emptyset$ (resp.,$\neg F^{-1}(0_U) = \emptyset$) or $F^{-1}(1_U)$ (resp.,$\neg F^{-1}(0_U)$) is a right ideal, a left ideal, and an ideal of Ξ, respectively.*

Proof. Suppose that $\mathfrak{F}$ is a fuzzy semibipolar soft right ideal over U and $F^{-1}(1_U) \neq \emptyset$. Let $\acute{\xi} \in F^{-1}(1_U)$ and $\grave{\xi} \in \Xi$. Then $F(\acute{\xi}) = 1_U$ Thus

$$F(\acute{\xi}\grave{\xi}) \succ F(\acute{\xi}) = 1_U.$$

Observe that $F(\acute{\xi}\grave{\xi}) = 1_U$. Thus $\acute{\xi}\grave{\xi} \in F^{-1}(1_U)$. Suppose $\grave{\xi} \leq_{\Xi} \acute{\xi}$. Then

$$F(\grave{\xi}) \succ F(\acute{\xi}) = 1_U.$$

It is true that $F(\grave{\xi}) = 1_U$. Hence $\grave{\xi} \in F^{-1}(1_U)$. It follows that $F^{-1}(1_U)$ is a right ideal of Ξ.

Assume that $\neg F^{-1}(0_U) \neq \emptyset$. Let $\check{\xi} \in \neg F^{-1}(0_U)$ and $\hat{\xi} \in \Xi$ be given. Then $\neg F(\check{\xi}) = 0_U$. Thus

$$\neg F(\check{\xi}\hat{\xi}) \prec \neg F(\check{\xi}) = 0_U.$$

Hence $\neg F(\check{\xi}\hat{\xi}) = 0_U$. Thus $\check{\xi}\hat{\xi} \in \neg F^{-1}(0_U)$. Suppose $\hat{\xi} \leq_\Xi \acute{\xi}$. Then

$$\neg F(\hat{\xi}) \prec \neg F(\check{\xi}) = 0_U.$$

Thus $\neg F(\hat{\xi}) = 0_U$. Therefore, $\hat{\xi} \in \neg F^{-1}(0_U)$. Consequently, $\neg F^{-1}(0_U)$ is a right ideal of Ξ. In a similar way, we can prove the other cases.

Proposition 2.27. *Let $(\Xi, *, \leq_\Xi)$ be an ordered groupoid and $\xi \in \Xi$. Then*

$$(F_{G^{-1}(1_U)}, \neg F_{\neg G^{-1}(0_U)}, \Xi) \Cap (G, \neg G, \Xi)$$

for all $(G, \neg G, \Xi) \in R_\xi$ (resp., L_ξ and I_ξ).

Proof. The proof is straightforward, so we omit it.

Lemma 2.28. *Let $(\Xi, *, \leq_\Xi)$ be an ordered groupoid and let Λ be a nonempty subset of Ξ. Then Λ is a right ideal (resp., a left ideal and an ideal) of Ξ if and only if the fuzzy semibipolar soft set $(F_\Lambda, \neg F_\Lambda, \Xi)$ over U concerning Λ is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal).*

Proof. Suppose that Λ is a right ideal of Ξ. Let $\acute{\xi}, \grave{\xi} \in \Xi$. Then, we consider the following four cases.

Case 1. Suppose $\acute{\xi}, \grave{\xi} \in \Lambda$. Then

$$F_\Lambda(\acute{\xi}) = 1_U = F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 0_U = \neg F_\Lambda(\grave{\xi}).$$

By the hypothesis, we have $\acute{\xi}\grave{\xi} \in \Lambda$. Hence $F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U$ and $\neg F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U$. Whence $F_\Lambda(\acute{\xi}\grave{\xi}) \succ F_\Lambda(\acute{\xi})$ and $\neg F_\Lambda(\acute{\xi}\grave{\xi}) \prec \neg F_\Lambda(\acute{\xi})$.

Case 2. Suppose $\acute{\xi}, \grave{\xi}$ are not elements of Λ. Then

$$F_\Lambda(\acute{\xi}) = 0_U = F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 1_U = \neg F_\Lambda(\grave{\xi}).$$

If $\acute{\xi}\grave{\xi} \in \Lambda$, then $F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U$ and $\neg F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U$, which yields

$$F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U \succ 0_U = F_\Lambda(\acute{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U \prec 1_U = \neg F_\Lambda(\acute{\xi}).$$

If $\acute{\xi}\grave{\xi} \notin \Lambda$, then $F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U$ and $\neg F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U$, which yields

$$F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U = F_\Lambda(\acute{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U = \neg F_\Lambda(\acute{\xi}).$$

Case 3. Suppose $\acute{\xi} \in \Lambda$ and $\grave{\xi} \notin \Lambda$. Then $\acute{\xi}\grave{\xi} \in \Lambda$. By the assumption, we obtain that $\acute{\xi}\grave{\xi} \in \Lambda$. Thus

$$F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U = F_\Lambda(\acute{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U = \neg F_\Lambda(\acute{\xi}).$$

Case 4. Suppose $\acute{\xi} \notin \Lambda$ and $\grave{\xi} \in \Lambda$. Then

$$F_\Lambda(\acute{\xi}) = 0_U = \neg F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 1_U = F_\Lambda(\grave{\xi}).$$

If $\acute{\xi}\grave{\xi} \in \Lambda$, then $F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U$ and $\neg F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U$, which yields

$$F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U \succ 0_U = F_\Lambda(\acute{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U \prec 1_U = \neg F_\Lambda(\acute{\xi}).$$

If $\acute{\xi}\grave{\xi} \notin \Lambda$, then $F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U$ and $\neg F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U$, which yields

$$F_\Lambda(\acute{\xi}\grave{\xi}) = 0_U = F_\Lambda(\acute{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}\grave{\xi}) = 1_U = \neg F_\Lambda(\acute{\xi}).$$

Next, we suppose that $\acute{\xi} \leq_\Xi \grave{\xi}$. Then, we consider the following four cases.

Case 1. Suppose $\acute{\xi}, \grave{\xi} \in \Lambda$. Then

$$F_\Lambda(\acute{\xi}) = 1_U = F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 0_U = \neg F_\Lambda(\grave{\xi}).$$

Case 2. Suppose $\acute{\xi}, \grave{\xi}$ are not elements of Λ. Then

$$F_\Lambda(\acute{\xi}) = 0_U = F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 1_U = \neg F_\Lambda(\grave{\xi}).$$

Case 3. Suppose $\acute{\xi} \in \Lambda$ and $\grave{\xi} \notin \Lambda$. Then

$$F_\Lambda(\acute{\xi}) = 1_U = \neg F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 0_U = F_\Lambda(\grave{\xi}).$$

Hence

$$F_\Lambda(\acute{\xi}) \succ F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) \prec \neg F_\Lambda(\grave{\xi}).$$

Case 4. Suppose $\grave{\xi} \in \Lambda$. By the hypothesis and $\acute{\xi} \in \Xi$, we have $\acute{\xi} \in \Lambda$. Then

$$F_\Lambda(\acute{\xi}) = 1_U = F_\Lambda(\grave{\xi}) \text{ and } \neg F_\Lambda(\acute{\xi}) = 0_U = \neg F_\Lambda(\grave{\xi}).$$

It follows that $(F_\Lambda, \neg F_\Lambda, \Xi)$ is a fuzzy semibipolar soft right ideal over U.

On the other hand, suppose that $(F_\Lambda, \neg F_\Lambda, \Xi)$ is a fuzzy semibipolar soft right ideal over U. Let $\acute{\xi} \in \Xi$ and $\grave{\xi} \in \Lambda$. Then

$$F_\Lambda(\grave{\xi}) = 1_U \text{ and } \neg F_\Lambda(\grave{\xi}) = 0_U.$$

By the assumption, we obtain that

$$F_\Lambda(\grave{\xi}\acute{\xi}) \succ F_\Lambda(\grave{\xi}) = 1_U \text{ and } \neg F_\Lambda(\grave{\xi}\acute{\xi}) \prec \neg F_\Lambda(\grave{\xi}) = 0_U.$$

Observe that $F_\Lambda(\grave{\xi}\acute{\xi}) = 1_U$ and $\neg F_\Lambda(\grave{\xi}\acute{\xi}) = 0_U$. Whence $\grave{\xi}\acute{\xi} \in \Lambda$. Next, assume $\acute{\xi} \leq_\Xi \grave{\xi}$. Then

$$F_\Lambda(\acute{\xi}) \succ F_\Lambda(\grave{\xi}) = 1_U \text{ and } \neg F_\Lambda(\acute{\xi}) \prec \neg F_\Lambda(\grave{\xi}) = 0_U.$$

We see that $F_\Lambda(\acute{\xi}) = 1_U$ and $\neg F_\Lambda(\acute{\xi}) = 0_U$. Thus $\acute{\xi} \in \Lambda$. This implies that Λ is a right ideal of Ξ.

Similarly, we can prove that other cases are also true.

Theorem 2.29. *Let $(\Xi, *, \leq_\Xi)$ be an ordered groupoid and $\xi \in \Xi$. Then, we have the following statements.*

(*i*) $R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi) = (F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$.

(*ii*) $L(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi) = (F_{L(\xi)}, \neg F_{L(\xi)}, \Xi)$.

(*iii*) $I(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi) = (F_{I(\xi)}, \neg F_{I(\xi)}, \Xi)$.

Proof. (*i*) By Notation 2.22 and Theorem 2.25, a fuzzy semibipolar soft set $(\widetilde{\bigcap}G, \widetilde{\bigcup}\neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to R_ξ is equal to the fuzzy semibipolar soft set $R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi)$ over U. From Remark 2.8, it follows that the fuzzy semibipolar soft set $(\widetilde{\bigcap}G, \widetilde{\bigcup}\neg G, \Xi)$ over U in which $(G, \neg G, \Xi)$ belongs to R_ξ is a fuzzy semibipolar soft subset of $(G, \neg G, \Xi)$ for all $(G, \neg G, \Xi) \in R_\xi$. Thus $R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi) \Subset (G, \neg G, \Xi)$ for all $(G, \neg G, \Xi) \in R_\xi$. Since $R(\xi)$ is a right ideal of Ξ, we have $(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$ is a fuzzy semibipolar soft right ideal over U due to Lemma 2.28. Note that $F_{R(\xi)}(\xi) = 1_U$ and $(\neg F_{R(\xi)})(\xi) = 0_U$. Then $(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi) \in R_\xi$. This means that $R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi) \Subset (F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$.

On the other hand, we shall prove that $(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$ is a fuzzy semibipolar soft subset of $(H, \neg H, \Xi)$ for all $(H, \neg H, \Xi) \in R_\xi$. Suppose $(H, \neg H, \Xi) \in R_\xi$. Then $H(\xi) = 1_U$ and $\neg H(\xi) = 0_U$. Therefore $\xi \in H^{-1}(1_U) \cap \neg H^{-1}(0_U)$. From Proposition 2.26, we have $H^{-1}(1_U)$ and $\neg H^{-1}(0_U)$ are right ideals of Ξ containing ξ. Thus $R(\xi) \subseteq H^{-1}(1_U)$ and $R(\xi) \subseteq \neg H^{-1}(0_U)$. From Remark 2.11 and Proposition 2.27, it follows that

$$(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi) \Subset (F_{H^{-1}(1_U)}, \neg F_{\neg H^{-1}(0_U)}, \Xi) \Subset (H, \neg H, \Xi).$$

Thus, we observe that a fuzzy semibipolar soft set $(\inf_\Xi\{H\}, \sup_\Xi\{\neg H\}, \Xi)$ over U in which $(H, \neg H, \Xi)$ belongs to R_ξ is a fuzzy semibipolar soft superset of $(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$. By Proposition 2.9, we see that

$$(\widetilde{\bigcap}H, \widetilde{\bigcup}\neg H, \Xi) = (\inf_\Xi\{H\}, \sup_\Xi\{\neg H\}, \Xi).$$

That is, we get that the fuzzy semibipolar soft set $(\widetilde{\bigcap}H, \widetilde{\bigcup}\neg H, \Xi)$ over U in which $(H, \neg H, \Xi)$ belongs to R_ξ is a fuzzy semibipolar soft superset of $(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$. By Notation 2.22 and Theorem 2.25, we obtain $(F_{R(\xi)}, \neg F_{R(\xi)}, \Xi) \Subset R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi)$. As a consequence, $R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi) = (F_{R(\xi)}, \neg F_{R(\xi)}, \Xi)$. Similarly, (ii) and (iii) follow.

Notation 2.30. *For an ordered groupoid $(\Xi, *, \leq_\Xi)$ and $\xi \in \Xi$, we denote*

$$R^B(\xi) := R(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi),$$

$$L^B(\xi) := L(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi),$$

$$I^B(\xi) := I(F_{\{\xi\}}, \neg F_{\{\xi\}}, \Xi).$$

Remark 2.31. *By Notations 2.22, 2.30 and Theorem 2.25, we observe that a fuzzy semibipolar soft set $(\widetilde{\bigcap}F, \widetilde{\bigcup}\neg F, \Xi)$ over U in which $(F, \neg F, \Xi)$ belongs to R_ξ (resp., L_ξ and I_ξ) is equal to $R^B(\xi)$(resp., $L^B(\xi)$ and $I^B(\xi)$). Then, by Proposition 2.14, we see that the following four statements hold.*

(i) $R^B(\xi) \in R_\xi$ *and* $R^B(\xi) \Subset (G, \neg G, \Xi)$ *for all* $(G, \neg G, \Xi) \in R_\xi$.

(ii) $L^B(\xi) \in L_\xi$ *and* $L^B(\xi) \Subset (G, \neg G, \Xi)$ *for all* $(G, \neg G, \Xi) \in L_\xi$.

(iii) $I^B(\xi) \in I_\xi$ *and* $I^B(\xi) \Subset (G, \neg G, \Xi)$ *for all* $(G, \neg G, \Xi) \in I_\xi$.

From Definition 2.10, it follows that $R^B(\xi)$ (resp., $L^B(\xi)$ and $I^B(\xi)$) is a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) over U containing ξ. This remark leads to the following definition.

Definition 2.32. *Let $(\Xi, *, \leq_\Xi)$ be an ordered groupoid and let $\xi \in \Xi$. In the following, we call $R^B(\xi)$ (resp., $L^B(\xi)$ and $I^B(\xi)$) a fuzzy semibipolar soft right ideal (resp., a fuzzy semibipolar soft left ideal and a fuzzy semibipolar soft ideal) over U generated by ξ.*

Definition 2.33. *Let $(\Xi, *, \leq_\Xi)$ be an ordered groupoid. We define relations $\mathcal{R}^B, \mathcal{L}^B$, and $\mathcal{I}^B$ on Ξ as follows:*

$$\mathcal{R}^B := \{(\acute{\xi}, \grave{\xi}) \in \Xi \times \Xi : R^B(\acute{\xi}) = R^B(\grave{\xi})\},$$

$$\mathcal{L}^B := \{(\acute{\xi}, \grave{\xi}) \in \Xi \times \Xi : L^B(\acute{\xi}) = L^B(\grave{\xi})\},$$

$$\mathcal{I}^B := \{(\acute{\xi}, \grave{\xi}) \in \Xi \times \Xi : I^B(\acute{\xi}) = I^B(\grave{\xi})\}.$$

Theorem 2.34. *Let* $(\Xi, *, \leq_\Xi)$ *be an ordered groupoid. Then*

$$\mathcal{R} = \mathcal{R}^B, \mathcal{L} = \mathcal{L}^B, \textit{and}\, \mathcal{I} = \mathcal{I}^B.$$

Proof. Let $\acute{\xi}, \grave{\xi} \in \Xi$ be given. Then, by Remark 2.11, Theorem 2.29, Notation 2.30, and Definition 2.33, we observe that

$$\begin{aligned}
(\acute{\xi}, \grave{\xi}) \in \mathcal{R}^B &\iff R^B(\acute{\xi}) = R^B(\grave{\xi}) \\
&\iff R(F_{\{\acute{\xi}\}}, G_{\{\acute{\xi}\}}, \Xi) = R(F_{\{\grave{\xi}\}}, G_{\{\grave{\xi}\}}, \Xi) \\
&\iff (F_{R(\acute{\xi})}, G_{R(\acute{\xi})}, \Xi) = (F_{R(\grave{\xi})}, G_{R(\grave{\xi})}, \Xi) \\
&\iff R(\acute{\xi}) = R(\grave{\xi}) \\
&\iff (\acute{\xi}, \grave{\xi}) \in \mathcal{R}.
\end{aligned}$$

Hence $\mathcal{R} = \mathcal{R}^B$. Similarly, we can prove that $\mathcal{L} = \mathcal{L}^B$ and $\mathcal{I} = \mathcal{I}^B$.

3 Summary

In this research, we introduced the concept of fuzzy semibipolar soft sets, defined as a special case of fuzzy bipolar soft sets in [10]. As a result we introduced the concept of fuzzy semibipolar soft right ideals, fuzzy semibipolar soft left ideals, and fuzzy semibipolar soft ideals in ordered groupoids. To describe Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ on ordered groupoids, we used the fuzzy semibipolar soft set as a tool of characterizations. We defined $\mathcal{R}^B, \mathcal{L}^B$ and $\mathcal{I}^B$ as relations in terms of fuzzy semibipolar soft right ideals, fuzzy semibipolar soft left ideals, and fuzzy semibipolar soft ideals, respectively. Then, we obtained that Green's relations $\mathcal{R}, \mathcal{L}$, and $\mathcal{I}$ can be described by $\mathcal{R}^B, \mathcal{L}^B$, and $\mathcal{I}^B$, respectively.

As studied above, a fuzzy semibipolar soft set can be one of the mathematical tools for describing the terminology of other classes in several algebraic structures. We observe that fuzzy semibipolar soft set theory in this work is one of the tools for studying uncertainty. In the future, we shall research the roughness induced by such a theory in terms of rough set theory [11, 12]. To find the optimal parameter and the best alternative element of a fuzzy semibipolar soft set, we plan to provide an algorithm for this process based on rough set theory.

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