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On a sequence attached to powerful numbers: an elementary proof

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Abstract

In this paper, we use a strictly elementary method to prove some results of Copil and Panaitopol [1] and we give a general form of their results.

1 Introduction

A natural number n is called powerfree if $p \mid n$, then $p^2 \nmid n$. A natural number n is called powerful if $p \mid n$, then $p^2 \mid n$. For every powerful number k, let c_k be the least positive integer such that kc_k is a square. In [1], Copil and Panaitopol studied some properties of the sequence of $(c_k)_{k\geq 1}$ and derived the following results:

$$\sum_{\substack{k\leq x \\ k \text{ powerful}}} c_k = \frac{3}{\pi^2}\zeta\left(\frac{4}{3}\right)x^{2/3} + O(x^{1/2}\log x), \tag{1.1}$$

$$\sum_{\substack{k\leq x \\ k \text{ powerful}}} c_k^2 = \frac{x}{3} + O(x^{5/6}), \tag{1.2}$$

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and

$$\sum_{\substack{k\le x\\ k \text{ powerful}}} \frac{1}{c_k} = \frac{\zeta\Big(\frac{5}{2}\Big)}{\zeta\Big(5\Big)} x^{1/2} + O(\log x). \tag{1.3}$$

In [3], Golomb proved that every powerful number can be written uniquely as n^2m^3, m is square-free. It follows immediately that c_k is its square-free part. We shall use this fact with the partial summation formula to give elementary proofs of (1.1)-(1.3). Moreover, we use this to generalize the formula (1.1)-(1.3). We obtain the following results:

Theorem 1.1. *For $\alpha > \frac{1}{2}$, we have the relation*

$$\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k^{\alpha} = \frac{6}{(\alpha+1)\pi^2}\zeta\Big(2(\alpha+1)/3\Big)x^{(\alpha+1)/3} + \begin{cases} O(x^{\alpha/3+1/6}), & \text{for } \alpha > 1; \\ O(x^{1/2}\log x), & \text{for } \alpha = 1; \\ O(x^{1/2}), & \text{for } \frac{1}{2} < \alpha < 1. \end{cases} \tag{1.4}$$

For $\alpha < \frac{1}{2}$, we have the relation

$$\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k^{\alpha} = \begin{cases} \frac{\zeta(3/2-\alpha)}{\zeta(3-2\alpha)} x^{1/2} + O(x^{(\alpha+1)/3}), & \text{for } \alpha < \frac{1}{2}; \\ \frac{6}{\pi^2}x^{1/2}\log x + O(x^{1/2}), & \text{for } \alpha = \frac{1}{2}. \end{cases} \tag{1.5}$$

2 Some preliminary Lemmas

We need the following Lemmas:

Lemma 2.1 (See in [2]). *For $x > 1$, we have*

$$Q(x) = \frac{6}{\pi^2}x + O(x^{1/2}),$$

where $Q(x)$ denotes the number of square-free integers $\le x$.

Lemma 2.2. *For $x > 1$, we have*

$$\sum_{m\le x} |\mu(m)| m^{\alpha} = \begin{cases} \frac{6}{(\alpha+1)\pi^2}x^{1+\alpha} + O(x^{\alpha+1/2}), & \text{for } \alpha > -1; \\ 6\pi^2 \log x + O(1), & \text{for } \alpha = -1; \\ \frac{\zeta(\alpha)}{\zeta(2\alpha)} + O(x^{1+\alpha}), & \text{for } \alpha < -1, \end{cases}$$

where $|\mu(m)|$ denotes the characteristic function of the set of square-free number.

Proof.
Using Lemma 2.1 and the partial summation, it follows that:

3 Proof of (1.1)-(1.3)

Proof of (1.1). Since every powerful number can be written uniquely as n^2m^3, m is square-free, we can write

$$\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k = \sum_{n^2m^3\le x} |\mu(m)|m = \sum_{n\le x^{1/2}} \sum_{m\le x^{1/3}n^{-2/3}} |\mu(m)|m.$$

In view of Lemma 2.2, we have

$$\begin{aligned}\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k &= \sum_{n\le x^{1/2}} \left(\frac{3}{\pi^2}x^{2/3}n^{-4/3} + O(x^{1/2}n^{-1})\right)\\ &= \frac{3}{\pi^2}x^{2/3}\sum_{n\le x^{1/2}} n^{-4/3} + O(x^{1/2}|\sum_{n\le x^{1/2}} n^{-1}|).\end{aligned} \tag{3.6}$$

Note that

$$\sum_{n\le x^{1/2}} n^{-4/3} = \zeta(4/3) + O(x^{-1/6}) \quad \text{and} \quad \sum_{n\le x^{1/2}} n^{-1} = O(\log x). \tag{3.7}$$

Then (1.1) follows from (3.6) and (3.7). □

Proof of (1.2). As in the proof of (1.1), we write:

$$\sum_{\substack{k\le x\\ k\text{powerful}}} c_k^2 = \sum_{n^2m^3\le x} |\mu(m)|m^2 = \sum_{n\le x^{1/2}} \sum_{m\le x^{1/3}n^{-2/3}} |\mu(m)|m^2.$$

In view of Lemma 2.2, we have

$$\begin{aligned}\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k^2 &= \sum_{n\le x^{1/2}} \left(\frac{2}{\pi^2}xn^{-2} + O(x^{5/6}n^{-5/3})\right)\\ &= \frac{2}{\pi^2}x\sum_{n\le x^{1/2}} n^{-2} + O(x^{5/6}|\sum_{n\le x^{1/2}} n^{-5/3}|).\end{aligned} \tag{3.8}$$

Note that

$$\sum_{n\le x^{1/2}} n^{-2} = \zeta(2) + O(x^{-1/2}) \quad \text{and} \quad \sum_{n\le x^{1/2}} n^{-5/3} = O(1). \tag{3.9}$$

Then (1.2) follows from (3.8), (3.9) and $\zeta(2) = \frac{\pi^2}{6}$. □

Proof of (1.3). As in the proof of (1.1), we write

$$\begin{aligned}\sum_{\substack{k\le x\\ k \text{ powerful}}} \frac{1}{c_k} &= \sum_{n^2m^3\le x} |\mu(m)|m^{-1} = \sum_{m\le x^{1/3}} |\mu(m)|m^{-1} \sum_{n\le x^{1/2}m^{-3/2}} 1\\ &= \sum_{m\le x^{1/3}} |\mu(m)|m^{-1}\Big(x^{1/2}m^{-3/2} + O(1)\Big)\\ &= x^{1/2}\sum_{m\le x^{1/3}} |\mu(m)|m^{-5/2} + O\Big(|\sum_{m\le x^{1/3}} m^{-1}|\Big).\end{aligned}$$

Then (1.3) follows from Lemma 2.2. □

4 Proof of Theorem 1.1

First, we prove (1.4) by using the same method as in the proof of (1.1). For $\alpha \ge 1$, we write

$$\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k^{\alpha} = \sum_{n^2m^3\le x} |\mu(m)|m^{\alpha} = \sum_{n\le x^{1/2}} \sum_{m\le x^{1/3}n^{-2/3}} |\mu(m)|m^{\alpha}.$$

In view of Lemma 2.2, we have

$$\begin{aligned}\sum_{\substack{k\le x\\ k \text{ powerful}}} c_k^{\alpha} &= \sum_{n\le x^{1/2}} \Big(\frac{1}{\alpha+1}\frac{6}{\pi^2}x^{2/3}n^{-2\alpha/3-2/3} + O(x^{\alpha/3+1/6}n^{-2\alpha/3-1/3})\Big)\\ &= \frac{1}{\alpha+1}\frac{6}{\pi^2}x^{2/3}\sum_{n\le x^{1/2}} n^{-2\alpha/3-2/3} + O(x^{\alpha/3+1/6}|\sum_{n\le x^{1/2}} n^{-2\alpha/3-1/3}|).\end{aligned} \tag{4.10}$$

From $\frac{2(\alpha+1)}{3} \ge \frac{4}{3}$, for $\alpha \ge 1$, we have

$$\sum_{n\le x^{1/2}} n^{-2\alpha/3-2/3} = \zeta\Big(\frac{2(\alpha+1)}{3}\Big) + O(x^{1/6-\alpha/3})$$

and (4.11)

$$\sum_{n\le x^{1/2}} n^{-2\alpha/3-1/3} = \begin{cases} O(1), & \text{for } \alpha > 1;\\ O(\log x), & \text{for } \alpha = 1.\end{cases}.$$

Then the first two cases of (1.4) follow from (4.10) and (4.11). The last case of (1.4) follows from (4.10) and the following fact:
For $\frac{1}{2} < \alpha < 1$,

$$\sum_{n \leq x^{1/2}} n^{-2\alpha/3-2/3} = \zeta\Big(\frac{2(\alpha+1)}{3}\Big) + O(x^{1/6-\alpha/3}) \ , \ \sum_{n \leq x^{1/2}} n^{-2\alpha/3-1/3} = O(x^{1/3-\alpha/3}).$$

For $\alpha \leq \frac{1}{2}$, we write

$$\begin{aligned} \sum_{\substack{k \leq x \\ k \text{ powerful}}} c_k^\alpha &= \sum_{n^2 m^3 \leq x} |\mu(m)| m^\alpha = \sum_{m \leq x^{1/3}} |\mu(m)| m^\alpha \sum_{n \leq x^{1/2} m^{-3/2}} 1 \\ &= \sum_{m \leq x^{1/3}} |\mu(m)| m^\alpha \Big(x^{1/2} m^{-3/2} + O(1) \Big) \\ &= x^{1/2} \sum_{m \leq x^{1/3}} |\mu(m)| m^{\alpha-3/2} + O\Big(| \sum_{m \leq x^{1/3}} m^\alpha | \Big). \end{aligned}$$

Then (1.5) follows from Lemma 2.2.

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