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On Rings Domination in Graphs

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Abstract

In this paper, we introduce a new parameter of domination in graphs called the inverse rings domination number. We determine the rings domination number and its inverse for complement of some certain graphs.

1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected and simple graph. A dominating set of the graph G is a set of vertices of the graph G, say D, such that each vertex in $V - D$ is adjacent to at least one vertex in D [1]. The concept of graph domination deals with various fields in graph theory as game theory [2] and [3], labeled graph [4], [5] and [6] topological graph [7], fuzzy graph [8], among others. In [9], the definition of domination is introduced under the condition that every vertex domination exactly two vertices. In [10], [11] and [12] the number of dominated vertices has been specified. For more types of domination number, see [13], [14], [15] and [16]. In [17], the rings domination number is introduced. A dominating set $D \subset V(G)$ in G is a rings domination if each vertex $v \in V - D$ is adjacent to at least two vertices in $V - D$. We present some fundamental results on

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rings domination are presented. Also, some operations on two graphs like a join, composition, cross product and corona, have been introduced and determined rings domination number for each of them. Here, the inverse rings domination is defined. We determine some results on rings domination for some complement graph and calculate Inverse rings domination for some certain graphs. For graph theoretic terminology we refer to Harary [18].

Definition 1.1. *[17] A dominating set $D \subset V(G)$ in G is a* **rings domination** *if each vertex $v \in V - D$ is adjacent to at least two vertices in $V - D$.*

Definition 1.2. *[17] Let $G = (V, E)$ be a graph. If D is a rings dominating set, then D is called a* **minimal rings dominating set** *if it has no proper rings dominating set. A minimum rings dominating set is a rings dominating set of smallest size in a given graph.*

Definition 1.3. *[17] The minimum cardinality of all minimal rings dominating set, denoted by $\gamma_{ri}(G)$, is called the* **rings domination number***.*

Observation 1.4. *[17] For a rings dominating set D of any graph G of order n, we have*

1. *The order of G is $n \geq 4$.*
2. *For each vertex $v \in V - D$, $deg(v) \geq 3$.*
3. $1 \leq |D| \leq n - 3$.
4. $3 \leq |V - D| \leq n - 1$.
5. $1 \leq \gamma_{ri} \leq |D| \leq n - 3$.

Proposition 1.5. *[17] Trees have no rings dominating set.*

Observation 1.6. *[17]*
1) $\gamma_{ri}(K_n) = \gamma_{ri}(W_n) = 1$, $n \geq 4$, $n \geq 5$ respectively.
2) S_n, C_n, and N_n has no rings dominating set.

Proposition 1.7. *[17] $\gamma_{ri}(k_{n,m}) = 2$, $n, m \geq 3$.*

2 Rings dominating set of complement of certain graphs

Theorem 2.1. *Let G be a path of order n. So $\gamma_{ri}(\overline{P_n}) = 2$, if $n \geq 6$, otherwise the graph $\overline{P_n} = 2$ has no rings dominating set.*

Proof. There are four cases as follows:
Case 1. If $n \leq 3$, then the complement of path graph has no rings dominating set according to Observation 1.4(1).
Case 2. If $n = 4$, then $\overline{P_n} \equiv P_4$; that is, this graph is self-complementary. As a result $\overline{P_4}$ according to Proposition 1.5.
Case 3. If $n = 5$, then the degree of each vertex in $\overline{P_5}$ which is not pendant vertex in graph P_5 is two. Thus, $\overline{P_5}$ has no rings dominating set according to Observation 1.4(2).
Case 4. If $n \geq 6$, then the vertices $\{v_1, v_n\}$ are of degree at least four and the other vertices are of degree at least three. It is clear that there is no rings dominating set with cardinality one since there is no vertex that is adjacent to all other vertices. Now, letting $D = \{v_1, v_2\}$, one can easily conclude that D is a minimum rings dominating set. Thus, $\gamma_{ri}(\overline{P_n}) = 2$. The proof is complete. □

Theorem 2.2. *Let G be a cycle of order n. So $\gamma_{ri}(\overline{C_n}) = 2$ if $n \geq 6$, otherwise the graph $\overline{C_n}$ has no rings dominating set.*

Proof. There are three cases as follows:
Case 1. If $n = 3, 4$, then each vertex in $\overline{C_n}$ are of degree less than or equal to one. Thus $\overline{C_n}$ has no rings dominating set according to Observation 1.4(2).
Case 2. If $n = 5$, then $\overline{C_n} \cong C_5$; that is, this graph is self-complementary. Again, each vertex in $\overline{C_n}$ is of degree two. Thus $\overline{C_n}$ has no rings dominating set according to Observation 1.4(2).
Case 3. If $n \geq 6$, then the graph $\overline{C_n}$ is $(n-3)$-regular. Thus the degree of each vertex is at least three. So let $D = \{v_i, v_j\}$, where the vertex v_i is adjacent to vertex v_j. The vertex v_i is adjacent to all vertices in the graph $\overline{C_n}$ except two vertices that are adjacent to it in the graph C_n. Thus, the set D is the minimum dominating set of the graph $\overline{C_n}$. Therefore, $\gamma_{ri}(\overline{C_n}) = 2$. From each case above, it is obvious that the proof is done. □

Theorem 2.3. *If G is a complete bipartite graph $K_{m,n}$, then*

$$\gamma_{ri}(\overline{K_{m,n}}) = \begin{Bmatrix} 2, if\ n = 1\ and\ m \geq 4\ or\ n, m \geq 4 \\ 3, \quad if\ n = 2\ and\ m \geq 4 \\ 4, \quad if\ n = 3\ and\ m \geq 4 \end{Bmatrix}$$ *otherwise the graph $\overline{K_{m,n}}$ has no rings dominating set.*

Proof. It is clear that $\overline{K_{m,n}} \equiv K_m \cup K_n$. Then there are three cases as follows:
Case 1.
Subcase 1. If $n = 1$ and $m \geq 4$, then it is clear that each minimum dominating set contains two elements one of them representing the component K_1 and the other representing component K_m. According to Observation 1.6(1), $\gamma_{ri}(\overline{K_{m,1}}) = 2$.
Subcase 2. If $n, m \geq 4$, then as before the graph $\overline{K_{m,n}}$ contains two components one of them is K_{ri} and the other is $K_{m,n}, m \geq 4$. Thus, $\gamma_{ri}(\overline{K_{m,1}}) = 2$ according to Observation 1.4(2).
Case 2. If $n = 2$ and $m \geq 4$, then as before the graph $\overline{K_{m,n}}$ contains two components one of them is K_2 and the other is K_m, $m \geq 4$. Thus, the two vertices of K_2 belong to every rings dominating set according to Observation 1.4(2). Secondly, since $m \geq 4$, it is enough to add one vertex from the component K_m to the dominating set. Thus, the minimum cardinality of all dominating set is three and the result obtains.
Case 3. If $n = 3$ and $m \geq 4$, then as before the graph $\overline{K_{m,n}}$ contains two components one of them is isomorphic to K_3 and the other is K_m, $m \geq 4$. Now, all vertices in the component K_3 belong to each rings dominating set. In the same manner as in Case 2, one vertex is enough to dominate the other component (K_m). Thus, $\gamma_{ri}(\overline{K_{m,n}}) = 4$.
Case 4. If $n, m \leq 3$, then each vertex in the graph $\overline{K_{m,n}}$ has degree less than three. Thus, the graph $\overline{K_{m,n}}$ has no rings dominating set according to Observation 1.4(2). The proof is now complete. □

Corollary 2.4. *If G is a star graph $S_n \cong K_{1,n-1}$, then $\gamma_{ri}(\overline{S_n}) = 2$ if $n \geq 5$, otherwise the graph $\overline{S_n}$ has no rings dominating set.*

Proposition 2.5. *Let G be a wheel graph of order n, $W_n \cong K_1 + C_{n-1}$. Then $\gamma_{ri}(\overline{W_n}) = 3$, if $n \geq 7$, otherwise the graph $\overline{W_n}$ has no rings dominating set.*

Proof. One can easily conclude that $\overline{W_n} \cong K_1 \cup \overline{C_{n-1}}$. Thus the result follows by Observation 1.4(2) and Theorem 2.3. □

Observation 2.6.
1) $\gamma_{ri}(\overline{N_n}) = \gamma_{ri}(K_n) = 1$, $n \geq 4$, otherwise the graph $\overline{N_n}$ has no rings dominating set.
2) $\overline{K_n}$ has no rings dominating set.

3 Inverse rings domination of certain graphs

Proposition 3.1. *If a graph G contains a vertex of degree less than or equal 2, then G has no inverse rings dominating set.*

Proof. It is straightforward from Observation 1.6(2). □

Observation 3.2. *1) The graphs P_n, S_n, C_n, and N_n have no inverse rings dominating set.*
2) ${\gamma_{ri}}^{-1}(K_n) = 1$, $n \geq 4$.

Proposition 3.3. *If G is a complete bipartite graph $K_{m,n}$, then ${\gamma_{ri}}^{-1}(K_{m,n}) = 2$, if $n, m \geq 4$, otherwise the graph G has no inverse rings dominating set.*

Proof. If $n, m \geq 4$, then it is clear that ${\gamma_{ri}}^{-1}(K_{m,n}) = 2$ by taking a rings dominating set that contains two vertices. One of them in the set V_1 and the other in the set V_2 such that these two vertices differ from the two vertices that were used in the rings dominating set of $K_{m,n}$. □

Proposition 3.4. *If G is a wheel graph W_n, then*
${\gamma_{ri}}^{-1}(W_n) = \left\{ \begin{matrix} \lceil \frac{n-1}{3} \rceil \; if\; n-1 \equiv 0, 1 (mod\, 3) \\ 1 + \lceil \frac{n-1}{3} \rceil \; if\; n-1 \equiv 0 (mod\, 3) \end{matrix} \right\}$, *$n \geq 5$, otherwise the graph G has no inverse rings dominating set.*

Proof. From Observation 1.6. $\gamma_{ri}(W_n) = 1$, $n \geq 5$. In this case the vertex that was used in the rings dominating set is the center of the wheel graph. Thus, to find a disjoint rings dominating set we must take the vertices from the induced subgraph isomorphic to the cycle of order n. There are two cases as follows:
Case 1. If $n-1 \equiv 0, 1 (mod 3)$, let the vertex set of the induced subgraph that isomorphic to cycle is $\{v_2, v_3, , v_n\}$ and let $D = \left\{ v_{2+3i}, i = 0, 1, ..., \lceil \frac{n-1}{3} \rceil - 1 \right\}$. It is clear that the set D is minimum dominating set to the wheel graph. Moreover, each vertex in the set $V - D$ has degree greater than or equal three. Thus, the set D is a minimum rings dominating set. Therefore, ${\gamma_{ri}}^{-1}(W_n) = \lceil \frac{n-1}{3} \rceil$.
Case 2. If $n-1 \equiv 2(mod 3)$, then again the set $D = \left\{ v_{2+3i}, i = 0, 1, , \lceil \frac{n-1}{3} \rceil - 1 \right\}$ is a minimum dominating set of the wheel graph. In this case, the vertex $v_n \in V - D$ and i is adjacent to only one vertex in the set $V - D$. Thus, the vertex must belong to the set D to satisfy the condition of a rings dominating set. Thus, ${\gamma_{ri}}^{-1}(W_n) = \lceil \frac{n-1}{3} \rceil + 1$. □

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