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Analytical Solution to a Quadratically Damped Quadratic Oscillator

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Abstract

In this work, we introduce a novel homotopy method for solving nonconservative oscillators. We illustrate the method to obtain approximate analytical solution to the quadratically and linearly damped Helmholtz quadratic oscillator. The obtained results are illustrated graphically and numerically.

1 Introduction

The equation of motion for the sinusoidally driven escape oscillator including nonlinear damping terms as a power series on the velocity reads [1]

$$\ddot{x}+2\varepsilon\,\dot{x}\,|\dot{x}|^{\lambda}+\omega_0^2x-\beta x^2=0,\ x(0)=x_0 \text{ and } x'(0)=\dot{x}_0 \text{ for } 0\leq t\leq T, \quad (1.1)$$

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where ε is the damping level, λ is the damping exponent. This is a power-law quadratically damped Helmholtz oscillator. In this paper, we give an analytical solution to this oscillator for any given arbitrary initial conditions. To this end, we make use of an improved homotopy method based on the Krylov-Bogoliubov-Mitropolsky method. First, for $\lambda \neq 0$, we approximate the term $\dot{x}\,|\dot{x}|^{\lambda}$ by means of a quintic Chebyshev polynomial, assuming that the magnitude of $\dot{x}$ is small. The approximation for $|\dot{x}| \leq M$ reads $\dot{x}\,|\dot{x}|^{\lambda} \approx r_0\dot{x} + r_1\dot{x}^3 + r_2\dot{x}^5$, where

$$\begin{aligned} r_0 &= \left(-\tfrac{1}{3}\left(\tfrac{169}{239}\right)^{\lambda} + \tfrac{41}{459}\left(\tfrac{482}{499}\right)^{\lambda} + \tfrac{13}{19}\left(\tfrac{5}{11}\right)^{1-\lambda} 2^{\lambda+2}17^{-\lambda}\right) M^{\lambda}. \\ r_1 &= \left(\tfrac{16}{3}\left(\tfrac{169}{239}\right)^{\lambda} - \tfrac{79}{209}\left(\tfrac{241}{499}\right)^{\lambda} 2^{\lambda+2} - \tfrac{123}{103}\left(\tfrac{11}{17}\right)^{\lambda} 2^{\lambda+4}5^{-\lambda-1}\right) M^{\lambda-2}. \\ r_2 &= \left(-\tfrac{16}{3}\left(\tfrac{169}{239}\right)^{\lambda} + \tfrac{1}{3}\left(\tfrac{11}{85}\right)^{\lambda} 2^{\lambda+3} + \tfrac{1}{3}\left(\tfrac{241}{499}\right)^{\lambda} 2^{\lambda+3}\right) M^{\lambda-4}. \end{aligned} \tag{1.2}$$

Next, we replace the original i.v.p. (1.1) with the i.v.p.

$$\ddot{x}+2\varepsilon\,\left(r_0\dot{x} + r_1\dot{x}^3 + r_2\dot{x}^5\right)+\omega_0^2 x-\beta x^2 = 0,\; x(0) = x_0 \text{ and } x'(0) = \dot{x}_0\;. \tag{1.3}$$

In the case when $\lambda = 0$ we have linear damping.

2 The Solution Method

Let $\lambda \neq 0$. We define the following homotopy:

$$H(x,p) = \ddot{x} + +\omega_0^2 x + p\left[2\varepsilon\,\left(r_0\dot{x} + r_1\dot{x}^3 + r_2\dot{x}^5\right) - \beta x^2\right] \tag{2.4}$$

and, following the idea of Krylov-Bogoliubov and Metropolsky (KBM), we seek an approximate anlaytical solution to the i.v.p. (1.3) in the equation form

$$x = x(t) = a(t)\cos(\psi(t)) + \sum_{n=1}^{N} p^n u_n(a(t),\psi(t)) + o\left(p^{N+1}\right), \tag{2.5}$$

where p stands for the homotopy parameter. The functions $a = a(t)$ and $\psi = \psi(t)$ obey the odes

$$a'(t) = \sum_{n=1}^{N} p^n A_n(a(t)), \psi'(t) = \omega_0 + \sum_{n=1}^{N} p^n \varphi_n(a(t)). \tag{2.6}$$

The functions u_n are chosen in order to avoid secularity terms. Once we have found these functions and the two functions $a = a(t)$ and $\psi = \psi(t)$, we set $p =$

1 and then the approximate analytical solution will be $x(t) = a(t)\cos(\psi(t)) + \sum_{n=1}^{N} u_n(a(t), \psi(t))$. In this paper, we perform the calculations for $N = 1$. Plugging equation (2.5) into (2.4) results in a large expression. The coefficient of p^1 is

$$p^1: \begin{array}{l} -\frac{1}{4}\omega_0\left(8a\varepsilon r_1 + 6a^3\varepsilon r_1\omega_0^2 + 5a^5\varepsilon r_2\omega_0^4 + 8A_1(a)\right)\sin(\psi) - 2a\omega_0\varphi_1(a)\cos(\psi) - \\ \frac{1}{2}a^2\beta\cos(2\psi) - \frac{1}{8}a^5\varepsilon\sin(5\psi)r_2\omega_0^5 + \\ \frac{1}{8}\left(4a^3\varepsilon r_1\omega_0^3 + 5a^5\varepsilon r_2\omega_0^5\right)\sin(3\psi) - \frac{a^2\beta}{2} + \omega_0^2\left(u_1(a,\psi) + {u_1}^{(0,2)}(a,\psi)\right). \end{array} \tag{2.7}$$

In order to avoid the appereance of secularity terms, the coefficients of $\sin(\psi)$ and $\cos(\psi)$ must be equal to zero so that

$$\varphi_1(a) = 0 \text{ and } A_1(a) = -\frac{1}{8}a\varepsilon\left(5a^4r_2\omega_0^4 + r_1\left(6a^2\omega_0^2 + 8\right)\right). \tag{2.8}$$

Replacing these expressions in (2.7) and equating the resulting expression to zero gives an ode for determining $u_1(a, \psi)$. Solving this ode gives

$$u_1(a,\psi) = \left(\begin{array}{c} c_1\cos(\psi) + \frac{1}{32}\left(32c_2 + 4a^3\varepsilon r_1\omega_0 + 5a^5\varepsilon r_2\omega_0^3\right)\sin(\psi) + \\ \frac{a^2\beta}{2\omega_0^2} - \frac{a^2\beta\cos(2\psi)}{6\omega_0^2} - \frac{1}{192}a^5\varepsilon r_2\omega_0^3\sin(5\psi) + \frac{1}{64}a^3\varepsilon\omega_0\left(4r_1 + 5a^2r_2\omega_0^2\right) \end{array}\right)\sin(3\psi), \tag{2.9}$$

where c_1 and c_2 are the constants of integration. Since secularity terms are not allowed, we set $c_1 = 0, c_2 = -1/32 \cdot a^3\varepsilon\omega_0\left(5a^2r_2\omega_0^2 + 4r_1\right)$. Then

$$u_1(a,\psi) = -\frac{1}{192}a^5\varepsilon r_2\omega_0^3\sin(5\psi) - \frac{a^2\beta\cos(2\psi)}{6\omega_0^2} + \frac{a^2\beta}{2\omega_0^2} + \frac{1}{64}a^3\varepsilon\omega_0\sin(3\psi)\left(5a^2r_2\omega_0^2 + 4r_1\right). \tag{2.10}$$

The odes for determining a and ψ are:

$$a'(t) = -\frac{5}{8}\varepsilon r_2\omega_0^4a^5(t) - \frac{3}{4}\varepsilon r_1\omega_0^2a^3(t) - \varepsilon r_1a(t). \tag{2.11}$$

$$\psi'(t) = \omega_0,\ \psi(t) = \omega_0t + B. \tag{2.12}$$

The final result is

$$\begin{array}{l} x(t) = a(t)\cos(\psi(t)) + \frac{\beta}{2\omega_0^2}a^2(t) - \frac{1}{192}\varepsilon r_2\omega_0^3\sin(5\psi(t))a^5(t) + \\ \left(\frac{5}{64}\varepsilon r_2\omega_0^3a^2(t) + \frac{1}{16}\varepsilon r_1\omega_0\right)a^3(t)\sin(3\psi(t)) - \frac{\beta}{6\omega_0^2}\cos(2\psi(t))a^2(t). \end{array} \tag{2.13}$$

Now, we must solve the ode (2.11). This ode cannot be evaluated in explicit form. However, using Chebyshev polynomials, we may use the following approximation:

$$-\tfrac{5}{8}\varepsilon r_2\omega_0^4 a^5(t) - \tfrac{3}{4}\varepsilon r_1\omega_0^2 a^3(t) - \varepsilon r_1 a(t) \approx$$
$$-\left(\varepsilon r_0 - \frac{5}{64}A^4\varepsilon r_2\omega_0^4\right)a(t) - \frac{1}{8}\varepsilon\left(6r_1\omega_0^2 + 5A^2 r_2\omega_0^4\right)a^3(t). \tag{2.14}$$

Then we solve the ode

$$a'(t) = -\left(\varepsilon r_0 - \frac{5}{64}A^4\varepsilon r_2\omega_0^4\right)a(t) - \frac{1}{8}\varepsilon\left(6r_1\omega_0^2 + 5A^2 r_2\omega_0^4\right)a^3(t),\ a(0) = A, \tag{2.15}$$

whose solution reads

$$a(t) = \frac{A\cdot\sqrt{1+S}}{\sqrt{1+S\exp(-\kappa t)}},\ S = -\frac{35A^4 r_2\omega_0^4 + 48A^2 r_1\omega_0^2 + 64r_0}{40A^4 r_2\omega_0^4 + 48A^2 r_1\omega_0^2},\ \kappa = 2\varepsilon r_0 - \frac{5}{32}A^4\varepsilon r_2\omega_0^4. \tag{2.16}$$

The constants A and B are determined from the initial conditions $x(0) = x_0$ and $x'(0) = \dot{x}_0$.

In the case when $\lambda = 0$, we have linear damping and then the problem reduces to

$$\ddot{x} + 2\varepsilon\dot{x} + \omega_0^2 x - \beta x^2 = 0,\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0 \text{ for } 0 \le t \le T, \tag{2.17}$$

This oscillator is integrable only when $\omega_0 = 2\varepsilon\sqrt{6}/5$. In this case, the exact solution to the i.v.p. (2.17) is written as

$$x(t) = \exp\left(-\frac{4\varepsilon}{5}t\right)y(f(t)), \tag{2.18}$$

where $\ddot{y} = \beta y^2$ and

$$y(0) = 0,\ y'(0) = \dot{x}_0,\ f(t) = \frac{5(1-\exp(-2\varepsilon/5t))}{2\varepsilon} \text{ if } x_0 = 0, \tag{2.19}$$

$$y(0) = \frac{1}{x_0},\ y'(0) = \frac{5\dot{x}_0 + 4\varepsilon x_0}{5x_0^3},\ f(t) = \frac{5(1-\exp(-2\varepsilon/5t))}{2\varepsilon}x_0 \text{ if } x_0 \neq 0. \tag{2.20}$$

The exact solution to the i.v.p. $\ddot{y} = \beta y^2$, $y(0) = y_0$ and $y'(0) = \dot{y}_0$ reads [2]

$$\begin{aligned} y &= c - \frac{3c}{1 - \frac{6}{c\beta}\wp\left(t \pm t_0; 0, \frac{c^3\beta^3}{54}\right)},\ c = \sqrt[3]{y_0^3 - \frac{3}{2\beta}\dot{y}_0^2}, \\ t_0 &= \wp^{-1}\left(-\frac{c(2c+y_0)\beta}{6(c-y_0)}; 0, \frac{c^3\beta^3}{54}\right). \end{aligned} \tag{2.21}$$

In general, the i.v.p. (2.17) does not admit an exact solution and then we solve it either by a numerical or an analytical method. In particular, we may solve it using the homotopy described above. Letting $N = 3$ in (2.5) gives the following solution:

$$\begin{gathered} x = x(t) = a\cos(\psi) + \frac{\left(\beta\left(4\varepsilon^2\cos(2\psi) - 12\varepsilon\sin(2\psi)\omega_0 + 9(3-\cos(2\psi))\omega_0^2\right)\right)}{54\omega_0^4}a^2 + \\ \frac{\left(\beta^2(17\varepsilon\sin(3\psi)+6\cos(3\psi)\omega_0)\right)a^3}{288\omega_0^5} - \frac{\left(\beta^3(-114+59\cos(2\psi)+\cos(4\psi))\right)a^4}{432\omega_0^6}. \end{gathered} \tag{2.22}$$

The odes for determining $a = a(t)$ and $\psi = \psi(t)$ are:

$$a'(t) = -\varepsilon a(t) - \frac{11\beta^2\varepsilon}{36\omega_0^4}a^3(t) \text{ and } \psi'(t) = \omega_0 - \frac{\varepsilon^2}{2\omega_0} - \frac{5\beta^2}{12\omega_0^3}a^2(t). \tag{2.23}$$

Their solutions are:

$$\begin{gathered} a(t) = \frac{6A\omega_0^2}{\sqrt{11A^2\beta^2(e^{2\varepsilon t}-1)+36\omega_0^4 e^{2\varepsilon t}}}. \\ \psi(t) = B - \frac{\varepsilon^2 t}{2\omega_0} + \frac{\omega_0}{22\varepsilon}\left(52\varepsilon t - 15\log\left(\frac{11A^2\beta^2\left(e^{2\varepsilon t}-1\right)}{36\omega_0^4} + e^{2\varepsilon t}\right)\right). \end{gathered} \tag{2.24}$$

The constants A and B are found from the initial conditions $x(0) = x_0$ and $x'(0) = \dot{x}_0$.

3 Analysis and Discussion

Let us analyze the accuracy of the obtained solution for different values.

Example 1. Let $0 \le t \le 100$, $\varepsilon = 0.25$, $\beta = 1$, $\omega_0 = 1$, $x_0 = 0$ and $\dot{x}_0 = 0.1$. The solution for $\lambda = 1.5$ and $M = 0.1$ is

$$\begin{gathered} x(t) = a\cos(\psi) + \\ 0.00130208a^2\left(91.2519a^3(\sin(5\psi) - 15\sin(3\psi)) + 46.7785a\sin(3\psi) - 64(\cos(2\psi)-3)\right), \\ a = a(t) = \frac{-0.0312961}{\sqrt{1.09803e^{0.00115273t}-1}},\ \psi = \psi(t) = t - 1.60452. \end{gathered} \tag{3.25}$$

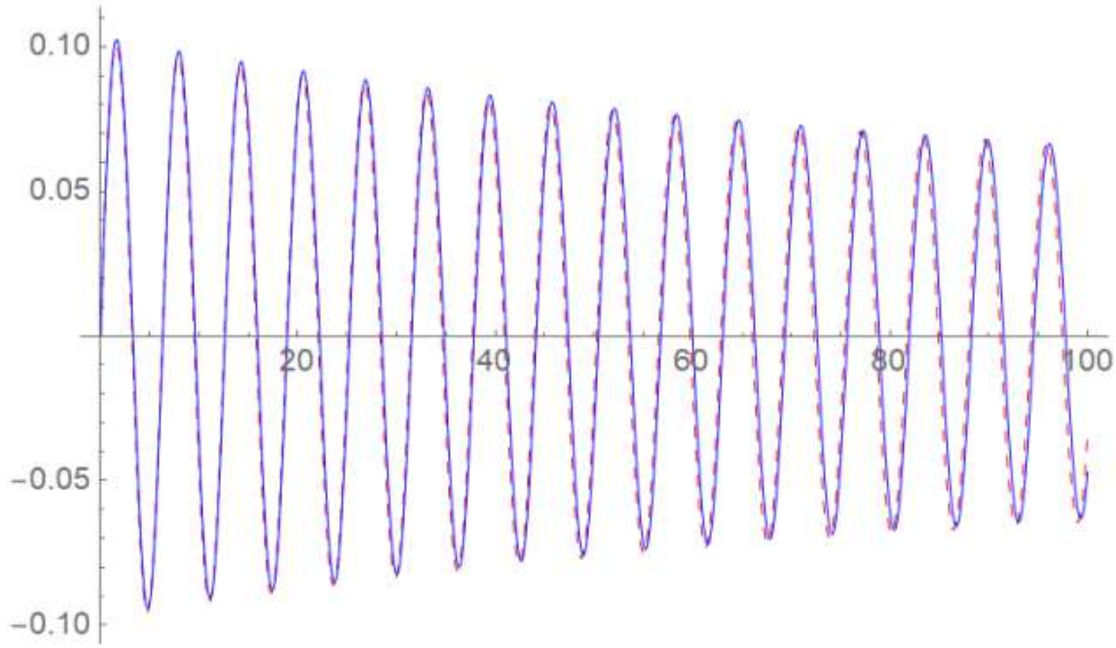

Figure 1. Comparison between the approximate and the numerical solution with $\lambda = 1.5$.The global error equals 0.0205607.

For other λ parameter values, the errors are depicted in Table 1.

λ	Error	λ	Error
0	7.50018×10^{-3}	1	8.27829×10^{-3}
0.1	6.7255×10^{-3}	1.1	9.62557×10^{-3}
0.2	5.77633×10^{-3}	1.2	1.15275×10^{-2}
0.3	5.71175×10^{-3}	1.3	1.375×10^{-2}
0.4	5.86775×10^{-3}	1.4	1.71553×10^{-2}
0.5	6.12634×10^{-3}	1.6	2.38375×10^{-2}
0.6	6.41615×10^{-3}	1.7	2.70068×10^{-2}
0.7	6.6984×10^{-3}	1.8	2.60342×10^{-2}
0.8	6.77398×10^{-3}	1.9	2.7899×10^{-2}
0.9	7.33952×10^{-3}	2	2.95259×10^{-2}

Table 1

Example 2. Let us consider the linearly damped Helmholtz oscillator

$$\ddot{x} + 0.5\dot{x} + x - \beta x^2 = 0,\ x(0) = 1 \text{ and } x'(0) = 0 \text{ for } 0 \le t \le T. \qquad (3.26)$$

For the parameter value $\beta = 0.5$ and $0 \le t \le T = 20$, the approximate analytical solution is evaluated using formula (2.22) for the data $A = 0.942$ and $B = -0.6359$. The error of the approximate analytical solution compared with the numerical one is 0.00647. See Figure 2.

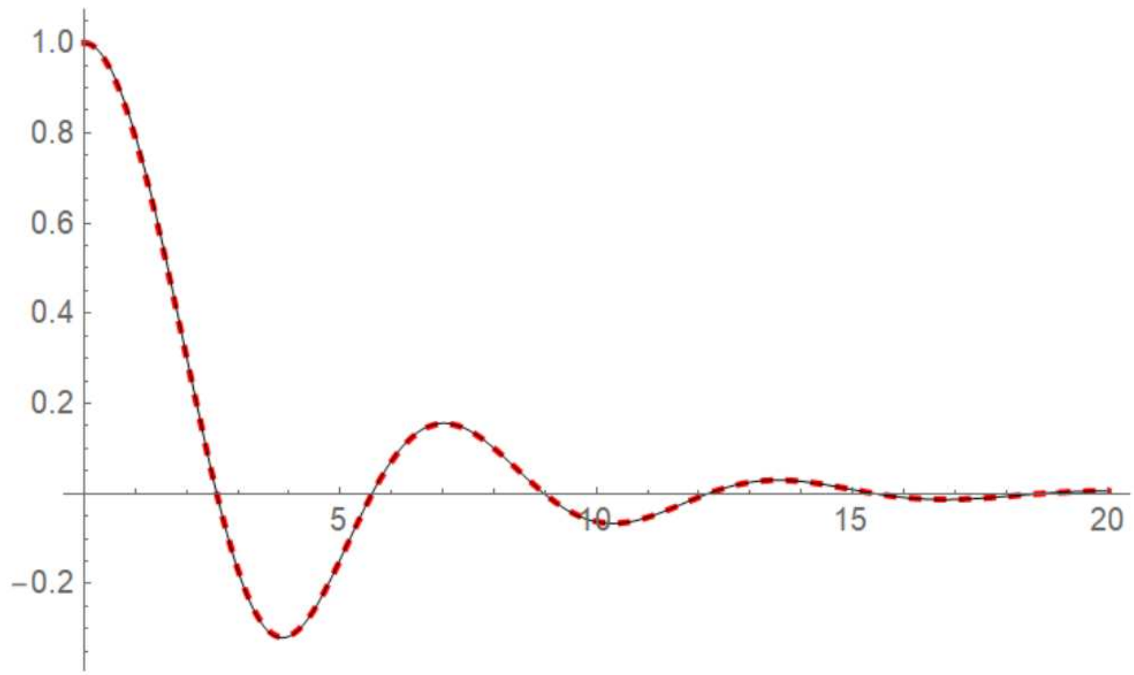

Figure 2.

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