International Journal of Mathematics and
Computer Science, **17**(2022), no. 3, 1145–1159

The Use of Two Newly Established Redlich-Kister Finite Differences with KSOR Method in a Numerical Solution of One Dimensional Telegraph Equations

Mohd Norfadli Suardi, Jumat Sulaiman

Faculty of Science and Natural Resources
Universiti Malaysia Sabah
Kota Kinabalu - 88400
Sabah, Malaysia

email: norfadli1412@gmail.com, jumat@ums.edu.my

(Received February 15, 2022, Accepted March 18, 2022)

Abstract

In this paper, we propose a numerical method to approximate the second order one dimensional linear hyperbolic telegraph equation which is based on two newly established second-order Redlich-Kister Finite Differences (RKFD) discretization scheme whose first two derivatives are used to obtain the second order RKFD approximation equation. This approximation leads to a linear system of the proposed problem where it will be solved using the Gauss-Seidel (GS) and Kaudd Successive Over Relaxation (KSOR) iterative methods. Then, three model examples are included to demonstrate the capability of the iterative methods considered for solving the proposed problem. In a numerical test problem, the performance for all numerical methods is shown by focusing on the number of iterations, execution time and maximum norm at five different mesh sizes. As results show, the KSOR method is found to be a better method in approximating the exact solution.

Key words and phrases: KSOR iteration, Redlich-Kister finite difference, Finite difference, Hyperbolic Telegraph problems.
AMS (MOS) Subject Classifications: 35L10, 65N06.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

1 Introduction

In recent years, there has been a need to develop, analyze and implement numerical solutions for linear partial differential equations (PDEs) due to the difficulty in obtaining their exact solutions [24, 29]. The hyperbolic telegraph equation is one of the PDEs that is actively used in solving the basis of mathematical problems in aerospace and industry, even in engineering, with its application applied in the vibration of structure and fundamental equation of atomic physics [5, 12]. Due to the difficulty in obtaining the hyperbolic solutions, many authors have worked on this problem by focusing on developing various methods in order to obtain a more accurate numerical solution of the hyperbolic telegraph equation. In [8], the author presented a numerical simulation of the hyperbolic equation by using the Sinc-Galerkin method. In [10], the authors presented a numerical solution of one dimensional telegraph equation by proposing the cubic B-spline collocation. In [25], the modification of cubic B-spline known as cubic trigonometric B-spline for solving one dimensional telegraph equation was used. Other numerical solutions of the hyperbolic equation can be seen in [26, 30, 13]. The numerical methods we listed before lead to proposing a new method called as Redlich-Kister Finite Difference (RKFD) method to solve the hyperbolic telegraph equation. This presented method is the basis of the Redlich-Kister function which ist important and useful in obtaining solutions in the fields of physics and chemistry fields but rarely used in other fields [28, 3, 22]. With the passage of time, this function started being applied to solve mathematical problems in numerical analysis. In numerical analysis, Hasan et al. [15] introduced a piecewise Redlich-Kister polynomial model and focused on constructing the first and third-order piecewse Redlich-Kister polynomial models and analyzed the relation between the Gauss-Seidel iteration and mesh sizes. As a result, they concluded that the third-order Redlich-Kister polynomial model is more accurate than the first-order Redlich-Kister model. More studies followed to explore more applications of the Redlich-Kister function in numerical analysis fields. For instance, Suardi and Sulaiman [17] presented a Redlich-Kister polynomial for solving the one dimensional boundary value problem. In addition, Suardi et al. [18, 19] proposed a Redlich-Kister finite difference (RKFD) which combine a Redlich-Kister polynomial and a finite difference method to solve two-point boundary value problems. Consequently, this served as a motivation to derive a numerical solution to the problem involving the one dimensional hyperbolic telegraph equation of the form

$$\frac{\partial^2 U}{\partial t^2}(x,t) + 2\alpha\frac{\partial U}{\partial t}(x,t) + \beta^2 U(x,t) = \frac{\partial^2 U}{\partial x^2}(x,t) + f(x,t), \tag{1.1}$$

with the initial condition

$$\begin{aligned} U(x,0) &= g_1(x), \\ \tfrac{\partial U}{\partial t}(x,0) &= g_2(x), \end{aligned}$$

and boundary conditions

$$\begin{aligned} U(0,t) &= g_3(t), \\ U(1,t) &= g_4(t), \end{aligned}$$

2 Redlich Kister Finite Difference Approximation Equation

In order to solve equation (1.1) numerically, the construction of two newly established RKFD approximation equations is needed as mentioned in the previous section. To construct this approximation equation, the proposed problem must go through the discretization process based on the RK function. Let us define the general formula of the RK function as

$$U_n(x,t) = \sum_{k=0}^{n} a_k(t).T_k(x), \tag{2.2}$$

where $a_k, k = 0, 1, 2, ..., n$ are to be calculated for the values of the unknown parameters considered.

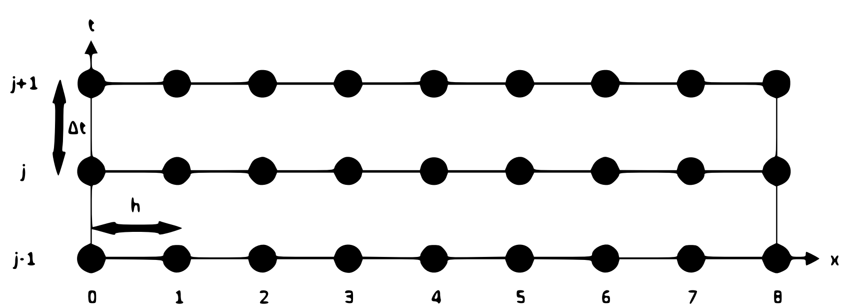

Figure 1: Distribution of mesh sizes at three levels considered.

Before starting on calculating the unknown parameters in equation (2.2), Figure 1 shows the distribution mesh sizes used until the discretization scheme process is completed. From the mesh sizes in Figure 1, we can construct the first three RK functions as shown in Figure 2.

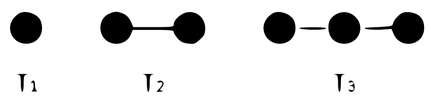

Figure 2: The path for T_1, T_2 and T_3

As a result of the first three RK functions being applied in equation (2.2), the RK approximation function can be written as

$$U(x,t) = a_0(t)T_0(x) + a_1(t)T_1(x) + a_2(t)T_2(x), \tag{2.3}$$

where the first three RK functions are defined as

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_2(x) = x(1-x).$$

Once again, looking back to Figure 1 and considering the node points $x_c = x_0 + ch, c = 0, 1, 2, ..., n$ in equation (2.3), the RK approximation function can form any group of three node points as the following equation shows

$$U_{c-1}(t) = a_0(t)T_{0,c-1} + a_1(t)T_{1,c-1} + a_2(t)T_{2,c-1}, \tag{2.4}$$

$$U_c(t) = a_0(t)T_{0,c} + a_1(t)T_{1,c} + a_2(t)T_{2,c}, \tag{2.5}$$

$$U_{c+1}(t) = a_0(t)T_{0,c+1} + a_1(t)T_{1,c+1} + a_2(t)T_{2,c+1}, \tag{2.6}$$

Next, we focus on calculating the unknown parameters in equation (2.3) by imposing the matrix approach to equations (2.4), (2.5) and (2.6). Substituting all the unknown parameters obtained into equation (2.3), we get

$$U(x,t) = N_0(x)U_{c-1}(t) + N_1(x)U_c(t) + N_2(x)U_{c+1}(t), \tag{2.7}$$

where the second-order RKFD shape functions are defined respectively as

$$\begin{aligned} N_0(x) &= \tfrac{1}{2h^2}(x^2 - 2xhc - xh + h^2c^2 + h^2c), \\ N_1(x) &= \tfrac{1}{h^2}(2xhc - x^2 - h^2c^2 + h^2), \\ N_2(x) &= \tfrac{1}{2h^2}(x^2 - 2xhc + xh + h^2c^2 - h^2c). \end{aligned} \tag{2.8}$$

and the first and second derivatives of RKFD shape function (2.8) are as follows

$$\begin{aligned} N_0^{'}(x) &= \tfrac{1}{2h^2}(2x-h-2hc), \\ N_1^{'}(x) &= \tfrac{1}{h^2}(2hc-2x), \\ N_2^{'}(x) &= \tfrac{1}{2h^2}(2x-h-2hc). \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} N_0^{''}(x) &= \tfrac{1}{h^2}, \\ N_1^{''}(x) &= \tfrac{-2}{h^2}, \\ N_2^{''}(x) &= \tfrac{1}{h^2}. \end{aligned} \tag{2.10}$$

Substituting equations (2.9) and (2.10) into equation (2.7), the first and second derivatives in equation (2.7) are given as

$$\frac{\partial U}{\partial x} = N_0^{'}(x_c)U_{c-1} + N_1^{'}(x_c)U_c + N_2^{'}(x_c)U_{c+1} \tag{2.11}$$

and

$$\frac{\partial^2 U}{\partial x^2} = N_0^{''}(x_c)U_{c-1} + N_1^{''}(x_c)U_c + N_2^{''}(x_c)U_{c+1}, \tag{2.12}$$

where $U(x_c) = U_c, c = 0, 1, 2, ..., n$ represent the approximation solution of function $U(x)$.The above expressions (2.11) and (2.12) represent the two newly established RKFD discretization scheme which led to this investigation as mentioned in the first section. Then, substitute equation (2.7) and its necessary derivatives into the equation (1.1), the RKFD approximation equation of the hyperbolic telegraph problem defined as

$$-\alpha_c U_{c-1,j+1} + \beta_c U_{c,j+1} - \gamma_c U_{c+1,j+1}, \tag{2.13}$$

where

$$\begin{aligned} \alpha_c &= (\triangle t)^2(N_0^{''}(x_c)), \\ \beta_c &= 1 + \alpha(\triangle t) + \beta^2(\triangle t)^2 - (\triangle t)^2(N_1^{''}(x_c), \\ \gamma_c &= (\triangle t)^2(N_2^{''}(x_c)), \\ R_{c,j+1} &= (\triangle t)^2 f_{c,j+1} + 2U_{c,j} + (\alpha\triangle t - 1)U_{c,j-1}, \end{aligned}$$

From the obtained equation (2.13), we continue to construct the linear system of the RKFD equation in matrix form as follows

$$W.U_{j+1} = R_{j+1}, j = 0, 1, 2, ..., n-1 \tag{2.14}$$

3 Derivation Of KSOR Iterative Method

In the previous section, the RKFD discretization scheme process led us to generate the large-scale and sparse linear system (2.14). When dealing with this linear system (2.14), the use of an iterative method is a suitable linear solver that need to be considered [7, 32, 31]. Accordinglyt, the KSOR iterative methods has been chosen to be tested in order to solve the linear system (2.14). In fact, the KSOR method is the extension of the SOR method [9, 23]. The approach used is by updating the current component to improve the numerical solution of the SOR method. Also, the implementation of this method will depend on the value of the weighted parameter, the optimum value chosen in this study. In order to formulate the KSOR method, recall the generated linear system (2.14) and replace its coefficient matrix by three summation matrices. This linear system becomes

$$(F + J + L).U = r, \tag{3.15}$$

where J is a diagonal matrix, F and L are triangular lower and upper matrices respectively.

Now, recalling and manipulating equation (3.15) to obtain the KSOR method in point iteration form, we get [9]

$$U^{(q+1)} = [(1-\omega)J - \omega F]^{-1}(J+L)U^{(q)} + [(1-\omega)J - \omega F]^{-1}R \tag{3.16}$$

where $U^{(q+1)}$ the indicates the current value U of at the $(q+1)^{th}$ iteration.

On the other hand, equation (3.16) can also be redefined as

$$U_c^{(q+1)} = \frac{1}{(1+\omega)}U_c^{(q)} + \frac{\omega}{(1+\omega)}(r_c - \alpha_c U_{c-1}^{(q+1)} - \gamma_c U_{c+1}^{(q)}), \tag{3.17}$$

where $c = 1, 3, 5, ..., n-1$. In addition, the optimum value of the weighted parameter needs to be considered during the implementation of the KSOR iterative method. A summary of the implementation of the KSOR iterative method for solving the one-dimensional telegraph problem is shown as Algorithm 1.

Algorithm 3.1. *KSOR iteration*
i) Set the initial value, $U = 0$.
ii) Calculate the coefficient matrix, W.
iii) Calculate the vector, R.

iv) Calculate equation (3.17).
v) Check the convergence test $\left|\left(\boldsymbol{U}^{(q+1)}\right)-\left(\boldsymbol{U}^{(q)}\right)\right| \leq \varepsilon = 10^{-10}$, *If yes, go to the next step. Otherwise, go back to step (iv).*
iv) Display approximate solution.

4 Numerical Problem and Discussion

In this section, the method derived and discussed in the previous section is used to perform a numerical testing for solving the proposed problem (1.1). Three model examples of the hyperbolic telegraph problems are included in this study and numerical results of the proposed method obtained are compared with GS iterative methods considered in term of the number of iterations (Iter), execution time (Time) and maximum norm (MaxNorm). The tolerance error used during this numerical testing was set ats $\varepsilon = 10^{-10}$.

Example 4.1. *[14] Consider the one-dimensional diffusion problem (1.1) with* $\alpha = 1$ *and* $\beta = 1$ *as*

$$\frac{\partial^2 U}{\partial t^2} + \alpha\frac{\partial U}{\partial t} + \beta U = \frac{\partial^2 U}{\partial x^2} + (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2e^{-t}, \qquad (4.18)$$

with the initial and boundary conditions

$$U(x,0) = 0,\ U(0,t) = 0,\ U(1,t) = 0,$$

and the analytical solution of problem (4.18) is $U(x,t) = (x - x^2)t^2e^{-t}$.

Example 4.2. *[4] Consider the one-dimensional diffusion problem (1.1) with* $\alpha = 4$ *and* $\beta = 2$ *as*

$$\frac{\partial^2 U}{\partial t^2} + \alpha\frac{\partial U}{\partial t} + \beta U = \frac{\partial^2 U}{\partial x^2} + (2 - \alpha + \beta)e^{-t}\sin(x), \qquad (4.19)$$

with the initial and boundary conditions are given as

$$U(x,0) = -\sin(x),\ U(0,t) = 0,\ U(1,t) = 0.$$

and the analytical solution of problem (4.19) is $U(x,t) = e^{-t}\sin(x)$.

Example 4.3. *[27] Consider the one-dimensional diffusion problem (1.1) with $\alpha = 1$ and $\beta = 1$ as*

$$\frac{\partial^2 U}{\partial t^2} + \alpha \frac{\partial U}{\partial t} + \beta U = \frac{\partial^2 U}{\partial x^2}, \tag{4.20}$$

with the initial and boundary conditions are given as

$$U(x,0) = -e^{x},\ U(0,t) = 0,\ U(1,t) = 0.$$

and the analytical solution of problem (4.19) is $U(x,t) = e^{x-t}$.

The numerical results of the GS and KSOR methods after the implementation of the numerical experiment are given in Table 1, 2 and 3.

A comparison of the numerical method by using two methods considered with all measuring parameters for five different mesh sizes as predicted in Table 1, 2 and 3 respectively. The numerical results here reported that the KSOR method is superior method compared to the GS method which it gives less iteration and time to approximate the known exact solution for each model examples considered. As can be observed at 256 mesh sizes from Table 1, the KSOR method improved the iteration which it needs 44 iteration compared 122 iteration for GS method. In the same table at the same mesh sizes, the KSOR needs the 0.09 seconds compared to 0.25 second for GS method. The improvement of the KSOR iteration highly improved the iteration and time when the big mesh sizes considered. For example the KSOR iteration needs the 579 iteration with 7.84 seconds for solving the problem (4.18) at n=4096 compared to the GS iterations. This is supported the earlier statement and it in line with the conclusion that made by [23]. For the numerical results for the problem (4.19) and (4.20) as depicted in Table 2 and 3, the same pattern of numerical results can be found as the problem (4.18) which show the KSOR iteration rapidly converged the problems considered in term of iteration and time compared to the GS iterations. For the accuracy comparison, all the iterative methods are converge their known exact solution very well.

5 Conclusion

In this paper, the development a numerical method based on two newly established RKFD discretization scheme with KSOR method for solving one dimensional hyperbolic Telegraph equation. In the proposed approach, the

Table 1: Numerical result for Example 1

Parameters	**Methods**	**n**				
		256	512	1024	2048	4096
Iter	GS	122	425	1528	5508	19692
	KSOR	44	84	159	303	579
Time	GS	0.25	1.28	6.70	43.86	310.61
	KSOR	0.09	0.20	0.59	2.15	7.84
MaxNorm	GS	7.1168e-04	7.1460e-04	7.2633e-04	7.7329e-04	9.6112e-04
	KSOR	7.1097e-04	7.1125e-04	7.1184e-04	7.1304e-04	7.1548e-04

Table 2: Numerical result for Example 2

Parameters	**Methods**	**n**				
		256	512	1024	2048	4096
Iter	GS	141	498	1818	6676	24403
	KSOR	59	114	224	437	856
Time	GS	0.18	1.71	8.72	52.36	379.93
	KSOR	0.11	0.25	0.90	3.16	11.65
MaxNorm	GS	9.8406e-04	9.8259e-04	9.7694e-04	9.5463e-04	8.6619e-04
	KSOR	9.8443e-04	9.8427e-04	9.8390e-04	9.8336e-04	9.8209e-04

Table 3: Numerical result for Example 3

Parameters	**Methods**	**n**				
		256	512	1024	2048	4096
Iter	GS	151	535	1963	7244	26640
	KSOR	63	121	237	464	910
Time	GS	0.31	1.45	8.41	58.28	411.05
	KSOR	0.15	0.26	0.89	3.30	12.29
MaxNorm	GS	2.0319e-04	2.0591e-04	2.1606e-04	2.5624e-04	4.1626e-04
	KSOR	2.0257e-04	2.0294e-04	2.0344e-04	2.0440e-04	2.0645e-04

first two derivatives of Redlich-Kister function are used during the discretization process to obtain the RKFD approximation equation which it will be leads to produce a linear system of the proposed problem. The obtained linear system is solved numerically by using the family of KSOR methods. To validate the performance of the KSOR method, three model examples are tested by implementing the Algorithm 1 and the numerical results being compared to the GS iterative method. From the numerical results produced, the two newly established RKFD discretization scheme with KSOR method are quite satisfactory in improving the iteration and time and has good agreement with the known exact solution. As further studies. the same discretization scheme in this paper can be extended to solve multi-dimensional boundary value problem with the two-step iteration family [1, 2], and the half-sweep [16, 6] and quarter-sweep [21, 11] approaches.

Acknowledgment. The authors would like to express their sincere gratitude to Universiti Malaysia Sabah for funding this research under UMSGreat research grant for a postgraduate student: GUG0494-1/2020.

References

[1] A. A. Dahalan, M. S. Muthuvalu, J. Sulaiman, Numerical solutions of two-point fuzzy boundary value problem using half-sweep alternating group explicit method, American Institute of Physics, **1557,** no. 1, (2013), 103–107.

[2] A. A. Dahalan, J. Sulaiman, M. S. Muthuvalu, Performance of HSAGE method with Seikkala derivative for 2-D fuzzy Poisson equation, Applied Mathematical Sciences, **8,** nos. 17-20, (2014), 885–899.

[3] A. Gayathri, T. Venugopal, K. Venkatramanan, Redlich-kister coefficients on the analysis of physico-chemical characteristics of functional polymers, Materials Today: Proceedings, **17,** (2019), 2083–2087.

[4] A. Kurt Bah, Y. Salih, A new algorithm for the numerical solution of telegraph equations by using Fibonacci polynomials. Mathematical and Computational Applications, **21,** no. 2, (2016), 15.

[5] A. Saadatmandi, M. Dehghan, Numerical solution of hyperbolic telegraph equation using the Chebyshev tau method, Numer. Methods Partial Differerential Equation, **26,** (2010), 239–252.

[6] A. Saudi, J. Sulaiman, Red-black strategy for mobile robot path planning, In World Congress on Engineering, **2182,** (2010), 2215–2219.

[7] D. M. Young, Iterative Solution Of Large Linear Systems, Academic Press, London, 1971.

[8] E. Hesameddini, E. Asadolahifard, The sinc-collocation method for solving the telegraph equation, Journal of Computer Engineering and Informatics, **1,** no. 1, (2013), 13–17.

[9] I. Youssef, On the successive overrelaxation method. Journal of Mathematics and Statistics, **8,** no. 2, (2012), 176–184.

[10] J. Rashidinia, S. Jamalzadeh, F. Esfahani, Numerical solution of one-dimensional telegraph equation using cubic B-spline collocation method, Journal of Interpolation and Approximation in Scientific Computing, (2014).

[11] J. V. Lung, J. Sulaiman, On quarter-sweep finite difference scheme for one-dimensional porous medium equations, International Journal of Applied Mathematics, **33,** no. 3, (2020), 439.

[12] M. Dehghan, A. Shokri, A numerical method for solving the hyperbolic telegraph equation, Numerical Methods Partial Differential Equation, **24,** (2008), 1080–1093.

[13] M. Dehghan, S. A. Yousefi, A. Lotfi, The use of Hes variational iteration method for solving the telegraph and fractional telegraph equations, Int. J. Numer. Methods Biomed. Engg., **27,** (2011), 219–231.

[14] M. Dugassa, F. Gemechis, A. Tesfaye, Numerical Solution of Second Order One Dimensional Linear Hyperbolic Telegraph Equation. Ethiopian Journal of Education and Sciences, **14,** no. 1, (2018), 39–52.

[15] M. K. Hasan, J. Sulaiman, S. Ahmad, M. Othman, S. A. Abdul Karim, Approximation of iteration number for gauss-seidel using redlich-kister polynomial, American Journal of Applied Sciences, **7,** (2010), 956– 962.

[16] M. K. Hasan, J. Sulaiman, S. A. Abdul Karim, M. Othman, Development of some numerical methods applying complexity reduction approach for solving scientific problem, (2010).

[17] M. N. Suardi, J. Sulaiman, Solution of One-Dimensional Boundary Value Problem by Using Redlich-Kister Polynomial. Lecture Notes in Electrical Engineering, **724,** (2021), 487–500.

[18] M. N. Suardi, J. Sulaiman, Redlich-Kister Finite Difference Solution for Solving Two-Point Boundary Value Problems by using Ksor Iteration Family. Advance Science Technology and Engineering System Journal, **6,** no. 1, (2021), 954–960.

[19] M. N. Suardi, J. Sulaiman, On Redlich-Kister finite difference solution of two-point boundary value problems using half-sweep kaudd successive over relaxation iteration. International Journal of Engineering Trends and Technology, **69,** no. 2, (2021), 77–82.

[20] M. Tatari, M. Haghighi, A generalized LaguerreLegendre spectral collocation method for solving initial-boundary value problems, Appl. Math. Model, **38,** (2014), 1351–1364.

[21] N. I. M. Fauzi, J. Sulaiman, Quarter-Sweep Modified SOR iterative algorithm and cubic spline basis for the solution of second order two-point boundary value problems, Journal of Applied Sciences (Faisalabad), **12,** no. 17, (2012), 1817–1824.

[22] N. P. Komninos, E. D. Rogdakis, Geometric investigation of the three-coefficient redlich-kister expansion global phase diagram for binary mixtures, Fluid Phase Equilibria, (2020), 112728.

[23] N. Z. F. M. Radzuan, M. N. Suardi, J. Sulaiman, KSOR iterative method with quadrature scheme for solving system of Fredholm integral equations of second kind, Journal of Fundamental and Applied Sciences, **9,** no. 5s, (2017), 609–623.

[24] R. Bernatz, Fourier series and numerical methods for partial differential equations, John Wiley and Sons, 2010.

[25] R. C. Mittal, R. Bhatia, Numerical Solution of second order hyperbolic telegraph equation via new cubic trigonometric Bsplines approach, Applied Mathematics and Computation, **220,** (2013), 496–506.

[26] R. Jiwari, S. Pandit, R. C. Mittal, A differential quadrature algorithm to solve the two dimensional linear hyperbolic telegraph equation with Dirichlet and Neumann boundary conditions, Appl. Math. Comput., **218,** (2012), 7279–7294.

[27] S. A. Yousefi, Legendre multiwavelet Galerkin method for solving the hyperbolic telegraph equation, Numerical Methods for Partial Differential Equations: An International Journal, **26,** no. 3, (2010), 535–543.

[28] S. Babu, R. Trabelsi, T. Srinivasa Krishna, N. Ouerfelli, A. Toumi, Reduced redlichkister functions and interaction studies of dehpa+ petrofin binary mixtures at 298.15 k, Physics and Chemistry of Liquids, **57,** no. 4, (2019), 536–546.

[29] S. S. Ray, Numerical analysis with algorithms and programming, Taylor and Francis Group, 2016.

[30] V. K. Srivastava, K. A. Mukesh, R. K. Chaurasia, Reduced differential transform method to solve two and three dimensional second order hyperbolic telegraphic equations, J. King Saud Univ. Eng. Sci., **17,** (2014).

[31] W. Hackbusch, Iterative Solution of Large Sparse Systems of Equations, Springer-Verlag, 1995.

[32] Y. Saad, Iterative Methods for Sparse Linear Systems, International Thomas Publishing, 1996.