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Generalized minimal Quasi-Ideals in LA-semigroups

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Abstract

The purpose of this paper is to study minimal quasi-ideals and minimal (m, n)-quasi-ideals in LA-semigroup.

1 Introduction and Preliminaries

The concepts of minimal quasi-ideals and minimal (m, n)-quasi-ideal of a semigroup was introduced by R.Chinaram [1]. The left almost semigroup (LA-semigroup) was first introduced by Kazin and Naseerudin [3].

Definition 1.1. [3] A groupoid $(S, \cdot)$ is called an *LA-semigroup* or an *AG-groupoid* if its satisfies left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in S$.

Definition 1.2. [3] An LA-semigroud S is called a *locally associative* LA-semigroup if its satisfies $(aa)a = a(aa)$ for all $a \in S$.

Lemma 1.3. *[5] In an LA-semigroup S its satisfies the* medial law *if* $(ab)(cd) = (ac)(bd)$ *for all* $a, b, c, d \in S$.

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Definition 1.4. [5] An element $e \in S$ is called *left identity* if $ea = a$ for all $a \in S$.

Lemma 1.5. *[3] If S is an LA-semigroup with left identity, then $a(bc) = b(ac)$ for all $a, b, c \in S$.*

Lemma 1.6. *[5] An LA-semigroup S with left identity its satisfies the* paramedial *if $(ab)(cd) = (dc)(ba)$ for all $a, b, c, d \in S$.*

Definition 1.7. [5] Let S be an LA-semigroup. A non-empty subset A of S is called an LA-subsemigroup of S if $AA \subseteq A$.

Definition 1.8. [**?**] A non-empty subset A of an LA-semigroup S is called a *left (right) ideal* of S if $SA \subseteq A(AS \subseteq A)$. As usual A is called an *ideal* if it is both left and right ideal.

Definition 1.9. [5] A non-empty subset A of an LA-semigroup S is called a *quasi ideal* of S if $SA \cap AS \subseteq A$.

Next we review, defined and study (m, n)-ideal and quasi ideal of LA-semigroup.

Definition 1.10. [2] A subset A of an LA-semigroup S is called

1. an (m, n)-*ideal* of S if $(A^m S)A^n \subseteq A$,

2. an (m, n)-*quasi ideal* of S if $S^m A \cap AS^n \subseteq A$,

where m and n are positive integers.

Lemma 1.11. *[4] Let S be an LA-semigroup and Q is an (m, n)-quasi ideal of S then $S^m Q$ and $S^n Q$ is an m-left ideal and n-right ideal of S respectively.*

Lemma 1.12. *Let S be an LA-semigroup and $a \in S$. Then the following statement hold.*

(1) $S^m a$ is an m-left ideal of S.

(2) aS^n is an n-left ideal of S.

(3) $S^m a \cap aS^n$ is am (m, n)-quasi ideal.

2 Minimal quasi-ideal in LA-semigroup

In section, we definition and study of minimal quasi-ideal in LA-semigroup is define the same as a minimal quasi-ideal in semigroup.

Definition 2.1. A quasi-ideal Q of an LA-semigroup S is called a *minimal quasi ideal* of S if Q dose not property contain any quasi ideal of S.

Theorem 2.2. *Let S be an LA-semigroup and Q a quasi-ideal of S. Then Q is a minimal quasi-ideal of S if and only if Q is the intersection of a minimal left ideal L and a minimal right ideal R of S.*

Proof. Assume that $Q = L \cap R$ for some a minimal left ideal L and a minimal right ideal R of S. So $Q \subseteq L$ and $Q \subseteq R$. Let Q' be a quasi-ideal of S contained in Q. Then $SQ' \subseteq SQ \subseteq SL \subseteq L$ and $Q'S \subseteq QS \subseteq RS \subseteq R$. By Lemma 1.11, SQ' and $Q'S$ are a left ideal and a right ideal of S, respectively. By the minimality of L and R, we have $SQ' = L$ and $Q'S = R$. Hence $Q = L \cap R = SQ' \cap Q'S \subseteq Q'$. Then $Q' = Q$. Therefore, Q is a minimal quasi-ideal of S.

Conversely, assume Q is a minimal quasi-ideal of S. Let $a \in Q$. Since aS and Sa are a right ideal and a left ideal of S respectively, we have $Sa \cap aS$ is a quasi-ideal of S. Since $Sa \cap aS \subseteq SQ \cap QS \subseteq Q$ we have by the minimality of Q. Then $Sa \cap aS = Q$. Next we shall show that Sa is a minimal left ideal of S. Let L be a left ideal of S contained in Sa. Then $L \cap aS \subseteq Sa \subseteq aS = Q$. Since $L \cap aS$ is a quasi-ideal of S we have the minimality of Q implies that $L \cap aS = Q$. Then $Q \subseteq L$. So $Sa \subseteq SQ \subseteq SL \subseteq L$. This implies that $L = Sa$. Thus the left ideal Sa is minimal. The minimality of the right ideal aS can be proved dually. □

Theorem 2.3. *Let Q be a quasi-ideal of an LA-semigroup S. Then the following statements are equivalent:*

(1) Q is a minimal quasi-ideal of S.

(2) $Q = Sa \cap aS$ for all $a \in Q$.

Proof. (1) $\Rightarrow$ (2) Assume Q is a minimal quasi-ideal of S. Let $a \in Q$. So $Sa \cap aS \subseteq SQ \cap QS \subseteq Q$. Since $Sa \cap aS$ is a quasi-ideal of S we have by the minimality of Q. Then $Q = Sa \cap aS$.

(2) $\Rightarrow$ (1) : Assume that $Q = Sa \cap aS$ for all $a \in Q$. Let Q' be a quasi-ideal of S contained in Q. Let $x \in Q'$ then $x \in Q$. By assumption, $Q = Sx \cap xS$. Therefore $Q = Sx \cap xS \subseteq SQ' \cap Q'S = Q'$. Hence $Q' = Q$. Thus Q is a minimal quasi-ideal of S. □

3 Minimal (m,n)-quasi-ideal in LA-semigroup

In section, we definition and study of minimal (m,n)-quasi-ideal in LA-semigroup is define the same as an minimal (m,n)-quasi-ideal in semigroup.

Definition 3.1. An (m,n)-quasi ideal of an LA-semigroup S is called a *minimal (m,n)-quasi ideal* of S if Q dose not property contain any (m,n)-quasi ideal of S. Minimal m-left ideals and minimal n-right ideals are defined analogously.

Theorem 3.2. *Let S be an LA-semigroup and Q an (m,n)-quasi-ideal of S. Then Q is minimal if and only if Q is the intersection of some minimal m -left ideal L and some minimal n-right ideal R of S.*

Proof. Assume that $Q = L \cap R$ for some minimal m-left ideal L and some minimal n-right ideal R of S. So $Q \subseteq L$ and $Q \subseteq R$. Let Q' be an (m,n)-quasi ideal of S contained in Q. Then $S^mQ \subseteq S^mQ \subseteq S^mL \subseteq L$ and $QS^n \subseteq QS^n \subseteq RS^n \subseteq R$. By Lemma 1.11 then S^mQ' and $Q'S^n$ is an m-left ideal and an n-right ideal of S, respectively. By the minimality of L and R, we have $S^mQ' \subseteq L$ and $Q'S^n \subseteq R$. Hence $Q = L \cap R = S^mQ' \cap Q'S^n \subseteq Q'$. Then $Q' = Q$. Therefore, Q is a minimal (m,n)-quasi ideal of S.

Conversely, assume that Q is a minimal (m,n)-quasi ideal of S. Let $a \in Q$. By Lemma 1.12, we have S^ma, aS^n and $S^ma \cap aS^n$ are an m-left ideal, an n-right ideal and an (m,n)-quasi-ideal of S, respectively. By the minimality of Q, since $S^ma \cap aS^n \subseteq S^mQ \cap QS^n \subseteq Q$, , $S^ma \cap aS^n = Q$.

Now, we shall show that S^ma is a minimal m -left ideal of S . Let L be an m -left ideal of S contained in S^ma. Then $L \cap aS^n \subseteq S^ma \cap aS^n = Q$. Since $L \cap aS^n$ is an (m,n)-quasi ideal of S, therefore the minimality of Q implies that $L \cap aS^n = Q$. Then $Q \subseteq L$. Therefore $S^ma \cap S^mQ \subseteq S^mL \subseteq L$. This implies $L = S^ma$. Thus the m -left ideal S^ma is minimal. The minimality of the n-right ideal aS^n can be proved dually. □

Corollary 3.3. *Let S be an LA-semigroup. Then S has at least one minimal (m,n)-quasi ideal if and only if S has at least one minimal m-left ideal and at least one minimal n-right ideal.*

Theorem 3.4. *Let S be an LA-semigroup. The following statements are true.*

(1) *An m-left ideal L is minimal if and only if $S^ma = L$ for all $a \in L$.*

(2) *An n-right ideal R is minimal if and only if $aS^n = R$ for all $a \in R$.*

(3) *An (m, n)-quasi ideal Q is minimal if and only if $S^m a \cap aS^n = Q$ for all $a \in Q$.*

Proof. (1) Assume that L is minimal. Let $a \in L$. Then $S^m a \subseteq S^m L \subseteq L$. By Lemma 1.12 (1), we have known that $S^m a$ is an m-left ideal of S. Since L is a minimal m-left ideal of S we have $S^m a = L$.

Conversely, assume that $S^m a = L$ for all $a \in L$. To show L is minimal, let L' be an m-left ideal of S contained in L. Let $x \in L'$ Then $x \in L$, by assumption. $S^m x = L$. Now we have $L = S^m x \subseteq S^m L' \subseteq L'$ Then $L \subseteq L'$. This implies that L is minimal.

(2) and (3) can be proved similarly to (1). □

Definition 3.5. [3] Let S be an LA-semigroup. S is called an (m, n)-quasi simple LA-semigroup if S is a unique (m, n)-quasi ideal of S. An m-left simple LA-semigroup and an n-right simple LA-semigroup are defined analogously.

Theorem 3.6. *Let S be an LA-semigroup. The following statements are true.*

(1) *S is an m-left simple LA-semigroup if and only if $S^m a = S$ for all $a \in S$.*

(2) *S is an n-right simple LA-semigroup if and only if $aS^n = S$ for all $a \in S$.*

(3) *Let Q be an (m, n)-quasi ideal of S. If Q is an (m, n)-quasi simple LA-semigroup, then Q is a minimal (m, n)-quasi ideal of S.*

Proof. (1) Since S is m-left simple, S is a minimal m-left ideal of S. By Theorem 3.4 (1), $S^m a = S$ for all $a \in S$.

Conversely, assume that $S^m a = S$ for all $a \in S$. By Theorem 3.4 (1), S is a minimal m-left ideal of S. Then S is an m-left simple LA-semigroup. (2) and (3) can be proved similarly to (1). □

Theorem 3.7. *Let S be an LA-semigroup. The following statements are true.*

(1) *S is an m-left of S. If L is an m-left simple LA-semigroup, then L is a minimal m-left ideal of S.*

(2) *S is an n-right of S. If R is an n-right simple LA-semigroup, then R is a minimal n-right ideal of S.*

(3) S is an (m,n)-quasi simple LA-semigroup if and only if $S^m a \cap aS^n$ for all $a \in S$.

Proof. (1) Let L be an m-left simple LA-semigroup. By Theorem 3.6 (1), $L^m a = L$ for all $a \in L$. For each $a \in L$ we have $L = L^m a \subseteq S^m a \subseteq S^m L \subseteq L$. Then $S^m a = L$ for all $a \in L$ By Theorem 3.4 (1) L is minimal.

(2) and (3) can be proved similarly to (1). □

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