International Journal of Mathematics and
Computer Science, **17**(2022), no. 3, 1033–1038

On Generalized (m, n)-bi-ideals in LA-semigroups

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(Received January 2, 2022, Accepted February 17, 2022)

Abstract

Akram [1] introduced the notion of (m, n)-ideal of LA-semigroups. The purpose of this paper is to define semiprime, prime and strongly prime generalized (m, n)-bi-ideal and study the relation among semiprime, prime and strongly prime generalized (m, n)-bi-ideal in LA-semigroups.

1 Preliminaries and basic definitions

The left almost semigroup (LA-semigroup) was first introduced by Kazin and Naseerudin [1].

Definition 1.1. [1] A groupoid $(S, .)$ is called an *LA-semigroup* or an *AG-groupoid* if it satisfies the left invertible law

$$(a.b).c = (c.b).a, \quad \text{for all} \quad a, b, c \in S.$$

Definition 1.2. [1] An LA-semigroup S is called a *locally associative* LA-semigroup if its satisfies

$$(aa)a = a(aa), \quad \text{for all} \quad a \in S.$$

Key words and phrases: LA-semigroup, ideal, (m, n)-bi-ideals, prime semiprime, strongly prime.
AMS (MOS) Subject Classifications: 16D25, 20M10, 20M99.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

Lemma 1.3. *[3] An LA-semigroup S satisfies the* medial law *if*

$$(ab)(cd) = (ac)(bd), \quad \textit{for all} \quad a, b, c, d \in S.$$

Definition 1.4. [5] An element $e \in S$ is called a *left identity* if $ea = a$, for all $a \in S$.

Lemma 1.5. *[1] If S is an LA-semigroup with a left identity, then*

$$a(bc) = b(ac), \quad \textit{for all} \quad a, b, c \in S.$$

Lemma 1.6. *[3] An LA-semigroup S with a left identity satisfies the* paramedial law *if*

$$(ab)(cd) = (dc)(ba), \quad \textit{for all} \quad a, b, c, d \in S.$$

Definition 1.7. [2] Let S be an LA-semigroup. A non-empty subset A of S is called an LA-subsemigroup of S if $AA \subseteq A$.

Definition 1.8. [2] A non-empty subset A of an LA-semigroup S is called a *left (right) ideal* of S if $SA \subseteq A(AS \subseteq S)$. As usual, A is called an *ideal* if it is both a left and a right ideal.

The notion of (m, n)-ideal of LA-semigroups was introduced by Akram [1]:

Definition 1.9. [1] A non-empty subset A of a semigroup S is called an (m, n)*-ideal* if A satisfies of relation

$$A^m S A^n \subseteq A,$$

where m, n are non-negative integers..

Definition 1.10. [2] Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be a *bi-ideal* of S if $(BS)B \subseteq B$.

The concept (m, n)-bi-ideal of an LA-semigroup was introduced by Gaketem

Definition 1.11. Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be (m, n)*-bi-ideal* of S if $(B^m S)B^n \subseteq B$, where m and n are arbitrary positive integers.

Note: The power B^m is canceled when $m = 0$; i.e., $(B^0 S) = S = (SB^0)$. Now we have the following definition:

Definition 1.12. Let S be an LA-semigroup. An LA-subsemigroup B of S is said to be $(m,0)$-*bi-ideal* of S if $(B^m S)B^0 \subseteq (B^m S) \subseteq B$ and $(0,n)$-*bi-ideal* of S if $(B^0 S)B^n \subseteq (SB^n) \subseteq B$.

In other words, we can say that an $(m,0)$-bi-ideal of S is exactly the m-left-ideal and the $(0,n)$-bi-ideal of S is exactly the n-right-ideal.

Here if $m = n = 1$, then B is called a generalized bi-ideal of S. If B is an LA-subsemigroup of S, then B is called an (m,n)-bi-ideal of S. If $m = n = 1$, then it is called a bi-ideal. We denote the collection of all generalized (m,n)-bi-ideals by $\overline{G(S)}$.

2 Main results

First, we define semiprime, prime, and strongly prime generalized (m,n)-bi-ideal,.

Definition 2.1. Let S be an LA-semigroup and let B be a (m,n)-bi-ideal. If $B \in \overline{G(S)}$, then B is called a semiprime generalized (m,n)-bi-ideal if $\forall K \in \overline{G(S)}, K^2 \subseteq B \Rightarrow K \subseteq B$.

The semiprime generalized (m,n)-bi-ideal of S is denoted by $\overline{SePG(S)}$

Definition 2.2. Let S be an LA-semigroup and let B be a (m,n)-bi-ideal. If $B \in \overline{G(S)}$, then B is called a prime generalized (m,n)-bi-ideal if $\forall K, L \in \overline{G(S)}, KL \subseteq B \Rightarrow K \subseteq B$ or $L \subseteq B$.

The prime generalized (m,n)-bi-ideal of S is denoted by $\overline{PG(S)}$

Definition 2.3. Let S be an LA-semigroup and let B be a (m,n)-bi-ideal. If $B \in \overline{G(S)}$, then B is called a strongly prime generalized (m,n)-bi-ideal if $\forall K, L \in \overline{G(S)}, \emptyset \neq KL \cap LK \subseteq B \Rightarrow K \subseteq B$ or $L \subseteq B$.

The strongly prime generalized (m,n)-bi-ideal of S is denoted by $\overline{StPG(S)}$

Theorem 2.4. *Suppose that $\overline{StPG(S)}$ denote a strongly prime, , $\overline{PG(S)}$ denote a prime, and $\overline{SePG(S)}$ denote the sets of semiprime generalized (m,n)-bi-ideal. Then we have the relation $\overline{StPG(S)} \subseteq \overline{PG(S)} \subseteq \overline{SePG(S)} \subseteq \overline{G(S)}$.*

Proof. Suppose that B is a strongly prime generalized (m,n)-bi-ideal of an LA-semigroup S. We must show that $\overline{StPG(S)} \subseteq \overline{PG(S)}$. Let $B \in \overline{StPG(S)}$. Since B is strongly prime, we have $C \subseteq B$ or $D \subseteq B$, $\forall C, D \in \overline{G(S)}$. Then B is prime. Thus $\overline{StPG(S)} \subseteq \overline{PG(S)}$.

Suppose that B is a prime generalized (m,n)-bi-ideal of an LA-semigroup S. We must show that $\overline{PG(S)} \subseteq \overline{SePG(S)}$. Let $C^2 \in B$. Then $CC \subseteq B$. Thus $C \subseteq B$. Therefore, B is semiprime.

Suppose that B is a semiprime generalized (m,n)-bi-ideal of an LA-semigroup S. We must show that $\overline{SePG(S)} \subseteq \overline{G(S)}$. Let $B \in \overline{G(S)}$. Since B is a semiprime, we have $B \in \overline{G(S)}$. So $\overline{SePG(S)} \subseteq \overline{G(S)}$.

Therefore, we have the relation $\overline{StPG(S)} \subseteq \overline{PG(S)} \subseteq \overline{SePG(S)} \subseteq \overline{G(S)}$. □

Theorem 2.5. *An arbitrary intersection of generalized (m,n)-bi-ideals is again a generalized bi-ideal.*

Proof. Since $\{A_i : i \in I\}$ is a family of generalized (m,n)-bi-ideal of an LA-semigroup S, the intersection of an LA-subsemigroup is an LA-subsemigroup. Next, we show that $B = \bigcap_{i=1}^{n} A_i$ is an (m,n)-bi-ideal of S. It suffices to prove that $(B^m S)B^n \subseteq B$. Let $x \in (B^m S)B^n$. Then $x = (b_1^m s)b_2^n$, for some $b_1^m, b_2^n \in B$ and $s \in S$. Thus, for any arbitrary $i \in I$, $b_1^m, b_2^n \in B_i$. So $x \in (B_i^m S)B_i^n$. Since B_i is a generalized (m,n)-bi-ideal of S, we have $(B_i^m S)B_i^n \subseteq B_i$. Then $x \in B_i$. Since i was chosen arbitrarily, $x \in B_i$ for all $i \in I$. Hence $x \in B$. So $(B^m S)B^n \subseteq B$. Hence $B = \bigcap_{i=1}^{n} A_i$ is an (m,n)-bi-ideal of S. □

Theorem 2.6. *An arbitrary intersection of (m,n)-bi-ideals is again a bi-ideal.*

Proof. The proof is similar to the one above with the obvious difference that $\{A_i : i \in I\}$ is a family of (m,n)-bi-ideal of an LA-semigroup S. □

Theorem 2.7. *Let S be an LA-semigroup. Then the following statements are equivalent:*

(1) $B^2 = B$, for every choice of generalized bi-ideal B from $\overline{G(S)}$ of S.

(2) $K \cap L = KL \cap LK$, for all bi-ideals K, L of S

(3) Each collection of generalized bi-ideals $\overline{G(S)}$ of S is semiprime

Proof. (1) ⇒ (2) Let K and L be any two generalized bi-ideals of an LA-semigroup S. Then, from our first hypothesis, we have

$$K \cap L = (K \cap L)^2 = (K \cap L)(K \cap L) \subseteq KL.$$

In a similar way, one can show that $K \cap L \subseteq LK$. Thus $(K \cap L) \subseteq KL \cap LK$.

Here both KL and LK are generalized bi-ideals as they are the product of two generalized bi-ideals. Moreover, $KL \cap LK$ is also a generalized bi-ideal. Therefore,

$$KL \cap LK = (KL \cap LK)^2 = (KL \cap KL)(KL \cap KL) \subseteq KL.LK \subseteq KSK \subseteq K.$$

Similarly, one can show that $KL \cap LK \subseteq L$. Thus $KL \cap LK \subseteq (K \cap L)$.

Hence $K \cap L = KL \cap LK$.

$(2) \Rightarrow (3)$ To show this assertion, we let K and B be any two generalized bi-ideal of S from the collection $\overline{G(S)}$ such that $K^2 \subseteq A$. Therefore, by hypothesis, we have $K = K \cap K = KK \cap KK = K^2$ and hence $K \subseteq B$. As a result, every generalized bi-ideal of S is semiprime.

$(3) \Rightarrow (1)$ Let B be a generalized bi-ideal of S from the collection $\overline{G(S)}$. If $B^2 = S$ (that is $B^2 = B$), then it is obvious that B is an idempotent. If $B^2 \neq S$, then B^2 is a proper generalized bi-ideal of S which contains B^2 and therefore, by hypothesis, $B^2 = \cap_a\{\overline{(G(S))}a : \overline{G(S)}$ is the collection of all the generalized bi-ideals$\}$. Since each $\overline{(G(S))}$ a is a semiprime generalized bi-ideal, $B \subseteq \overline{(G(S))}_a$, for all a and so $B \subseteq \overline{(G(S))}_a = B^2$. Hence each generalized bi-ideal in S is idempotent; i.e., $B^2 = B$. □

Theorem 2.8. *Let $ab \in \{a,b\}S\{a,b\}$, for every choice of a and b in an LA-semigroup S. Then every generalized bi-ideal of S is a bi-ideal of S.*

Proof. Let B be a generalized bi-ideal. Then B is a nonempty subset such that $BSB \subseteq B$.

Now, we want to show that B is an LA-subsemigroup. Let $a, b \in B$. Then, as $ab \in \{a,b\}S\{a,b\}$ and $\{a,b\} \subseteq A$, we have $ab \in \{a,b\}S\{a,b\} \subseteq BSB \subseteq B$. So B is an LA-subsemigroup and hence is a bi-ideal of S. □

Theorem 2.9. *Let $ab \in \{a,b\}S\{a,b\}$, for every choice of a and b in an LA-semigroup S. Then every generalized (m,n) bi-ideal of S containing the zero element is an (m,n)-bi-ideal of S.*

Proof. Let B be a generalized (m,n)-bi-ideal of S. Then B is non-empty subset such that $B^m S B^n \subseteq B$. We want to show that B is an LA-subsemigroup. Let $a, b \in B$. Then $ab \in \{a,b\}S\{a,b\}$. Now, $a, b \subseteq S$ implies that $\{a,b\} \subseteq B^m$ since B has the zero element. Similarly, $\{a,b\} \subseteq B^n$. Therefore, $ab \in B^m S B^n \subseteq B$ and hence B is an (m,n)-bi-ideal of S. □

Theorem 2.10. *Let S an LA-semigroup. Suppose that every generalized (m,n)-bi-ideal of S is an (m,n)-bi-ideal of S. Then $a^m b^m \in \{a^m, b^n\}S\{a^m, b^n\}$, for every choice of a and b in an LA-semigroup S.*

Proof. Suppose $a, b \in S$. Now, consider the generalized (m,n)-bi-ideal generated by the subset $\{a^m, b^n\}$ of S. By definition, it is the set B such that $B = \cap_{a \in I} N_a$, where N_a is a generalized (m,n)-bi-ideal containing a^m, b^n. We claim that $B = \{a^m, b^n\} \cup \{a^m, b^n\}S\{a^m, b^n\}$. Clearly, B contains $\{a^m, b^n\}$.

Now, suppose that N is some other generalized (m,n)-bi-ideal containing $\{a^m, b^n\}$. To prove our claim, it is sufficient to show that $A \subseteq N$. Let $x = a^m$, for some $s \in S$. This is because if $x = a^m$ or $x = b^m$, then clearly $x \in N$. Now, as N is a generalized (m,n)-bi-ideal, $N^mSN^n \subseteq N$, $a^msb^n \subseteq N$. Hence, $x \in N$ and $B \subseteq N$.

Now, $\{a^m, b^n\} \cup \{a^m, b^n\}S\{a^m, b^n\}$ is also a generalized (m,n)-bi-ideal containing $\{a^m, b^n\}$. It follows that $B = \{a^m, b^n\} \cup \{a^m, b^n\}S\{a^m, b^n\}$. Since every generalized (m,n)-bi-ideal of S is an (m,n)-bi-ideal of S, a^mb^n is a generated element of the set $\{a^m, b^n\}$ and so $a^m, b^n \in B$. As a^m, b^n does not belong to $\{a^m, b^n\}$ $a^m, b^n \in \{a^m, b^n\}S\{a^m, b^n\}$. This completes the proof. $\square$

Acknowledgement. The author is very grateful to the anonymous referee for stimulating comments and improving the presentation of this paper.

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