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Weak quasi (Λ, sp)-continuity for multifunctions

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Abstract

The main goal of this paper is to introduce the concept of weakly quasi (Λ, sp)-continuous multifunctions. Moreover, we investigate several characterizations of weakly quasi (Λ, sp)-continuous multifunctions.

1 Introduction

In 1961, Marcus [7] introduced and studied the concept of quasi continuous functions. Neubrunnová [8] showed that quasi continuity is equivalent to semi-continuity due to Levine [5]. In 1973, Popa and Stan [12] introduced and investigated the concept of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [6] which are independent of each other. In 1975, Bânzaru and Crivăţ [3] introduced and studied the concept of quasi continuous multifunctions. In 1988, Noiri and Popa [10] introduced and investigated the notion of weakly quasi continuous multifunctions. Some properties of weakly quasi continuous multifunctions have been obtained in [12]. In 2002, Popa and Noiri [11] investigated the

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further characterizations of weakly quasi continuous multifunctions by using preopen sets, semi-open sets, β-open sets and α-open sets. Abd El-Monsef et al. [1] introduced a weak form of open sets called β-open sets. In 2004, Noiri and Hatir [9] introduced the notion of Λ_{sp}-sets in terms of the concept of β-open sets and investigated the notion of Λ_{sp}-closed sets by using Λ_{sp}-sets. In [2], the author introduced the concepts of (Λ, sp)-open sets and (Λ, sp)-closed sets which are defined by utilizing the notions of Λ_{sp}-sets and β-closed sets. The purpose of the present paper is to introduce the notion of weakly quasi (Λ, sp)-continuous multifunctions. Moreover, some characterizations of weakly quasi (Λ, sp)-continuous multifunctions are discussed.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure and the interior of A, respectively. A subset A of a topological space (X, τ) is said to be β-open [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called β-closed. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [10] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$. A subset A of a topological space (X, τ) is called a Λ_{sp}-set [10] if $A = \Lambda_{sp}(A)$. A subset A of a topological space (X, τ) is called (Λ, sp)-*closed* [2] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called (Λ, sp)-*open*.

Let A be a subset of a topological space (X, τ) and let $x \in X$. A point $x \in X$ is called a (Λ, sp)-*cluster point* [2] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the (Λ, sp)-*closure* [2] of A and is denoted by $A^{(\Lambda,sp)}$. The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)-*interior* [2] of A and is denoted by $A_{(\Lambda,sp)}$. A subset A of a topological space (X, τ) is said to be $s(\Lambda, sp)$-*open* (resp. $p(\Lambda, sp)$-*open*, $\beta(\Lambda, sp)$-*open*, $r(\Lambda, sp)$-*open*) if $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$ (resp. $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$, $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$, $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$) [2]. The complement of a $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) set is said to be $s(\Lambda, sp)$-*closed* (resp. $p(\Lambda, sp)$-*closed*, $\beta(\Lambda, sp)$-*closed*, $r(\Lambda, sp)$-*closed*). The family of all $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $r(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$ (resp. $p\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$, $r\Lambda_{sp}O(X, \tau)$).

Let A be a subset of a topological space (X, τ). The intersection of all $s(\Lambda, sp)$-closed sets containing A is called the $s(\Lambda, sp)$-*closure* of A and is denoted by $A^{s(\Lambda,sp)}$. The union of all $s(\Lambda, sp)$-open sets contained in A is called the $s(\Lambda, sp)$-*interior* of A and is denoted by $A_{s(\Lambda,sp)}$.

Following [4], by a multifunction $F : (X, \tau) \to (Y, \sigma)$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$ and for each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y. For any (Λ, sp)-open set V of a topological space (Y, σ), we denote $V^+ = \{B \in \mathcal{P}(Y) \mid B \subseteq V\}$ and $V^- = \{B \in \mathcal{P}(Y) \mid B \cap V \neq \emptyset\}$.

3 Weakly quasi (Λ, sp)-continuous multifunctions

In this section, we introduce the notion of weakly quasi (Λ, sp)-continuous multifunctions. In particular, several characterizations of weakly quasi (Λ, sp)-continuous multifunctions are discussed.

Definition 3.1. *A multifunction* $F : (X, \tau) \to (Y, \sigma)$ *is said to be weakly quasi* (Λ, sp)*-continuous if, for each* $x \in X$*, each* (Λ, sp)*-open neighborhood* V *of* x *and any* (Λ, sp)*-open sets* G_1, G_2 *of* Y *such that* $F(x) \in G_1^+ \cap G_2^-$*, there exists a nonempty* (Λ, sp)*-open set* U *of* X *such that* $U \subseteq V$*,* $F(U) \subseteq G_1^{(\Lambda,sp)}$ *and* $F(z) \cap G_2^{(\Lambda,sp)} \neq \emptyset$ *for every* $z \in U$*.*

Theorem 3.2. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is weakly quasi* (Λ, sp)*-continuous;*

(2) for each $x \in X$ *and any* (Λ, sp)*-open sets* G_1, G_2 *of* Y *such that* $F(x) \in G_1^+ \cap G_2^-$*, there exists a* $s(\Lambda, sp)$*-open set* U *of* X *containing* x *such that* $F(U) \subseteq G_1^{(\Lambda,sp)}$ *and* $F(z) \cap G_2^{(\Lambda,sp)} \neq \emptyset$ *for every* $z \in U$*;*

(3) $[[F^-([K_1]_{(\Lambda,sp)}) \cap F^+([K_2]_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$ *for every* (Λ, sp)*-closed sets* K_1, K_2 *of* Y*;*

(4) $F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)}$ *for every* (Λ, sp)-*open sets* V_1, V_2 *of* Y*;*

(5) $[F^+(V_1) \cup F^-(V_2)]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cap F^+(V_2^{(\Lambda,sp)})$ *for every* (Λ, sp)-*open sets* V_1, V_2 *of* Y*.*

Proof. (1) ⇒ (2): Let $\mathcal{U}(x)$ be the family of all (Λ, sp)-open sets of X containing x. Let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. For each $H \in \mathcal{U}(x)$, there exists a nonempty (Λ, sp)-open set G_H such that $G_H \subseteq H$, $F(G_H) \subseteq V_1^{s(\Lambda,sp)}$ and $F(y) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $y \in G_H$. Let $W = \cup\{G_H \mid H \in \mathcal{U}(x)\}$. Then, W is (Λ, sp)-open in X, $x \in W^{(\Lambda,sp)}$, $F(W) \subseteq V_1^{s(\Lambda,sp)}$ and $F(w) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq W^{(\Lambda,sp)}$. Thus, U is a $s(\Lambda, sp)$-open set of X containing x such that $F(U) \subseteq V_1^{s(\Lambda,sp)}$ and $F(z) \cap V_2^{s(\Lambda,sp)} \neq \emptyset$ for every $z \in U$.

(2) ⇒ (4): Let V_1, V_2 be any (Λ, sp)-open sets of Y and let

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then, $F(x) \in V_1^+ \cap V_2^-$ and there exists a $s(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V_1^{(\Lambda,sp)}$ and $F(z) \cap V_2^{(\Lambda,sp)} \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{(\Lambda,sp)}$ and hence

$$F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{(\Lambda,sp)}.$$

(4) ⇒ (5): Let V_1, V_2 be any (Λ, sp)-open sets of Y. Then by (4), we have

$$\begin{aligned}
&X - [F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})] \\
&= [X - F^-(V_1^{(\Lambda,sp)})] \cap [X - F^+(V_2^{(\Lambda,sp)})] \\
&= F^+(Y - V_1^{(\Lambda,sp)}) \cap F^-(Y - V_2^{(\Lambda,sp)}) \\
&\subseteq [F^+([Y - V_1^{(\Lambda,sp)}]^{(\Lambda,sp)}) \cap F^-([Y - V_2^{(\Lambda,sp)}]^{(\Lambda,sp)})]_{s(\Lambda,sp)} \\
&= [F^+(Y - [V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cap F^-(Y - [V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{s(\Lambda,sp)} \\
&\subseteq [F^+(Y - V_1) \cap F^-(Y - V_2)]_{s(\Lambda,sp)} \\
&= [[X - F^-(V_1)] \cap [X - F^+(V_2)]]_{s(\Lambda,sp)} \\
&= [X - (F^-(V_1) \cup F^+(V_2))]_{s(\Lambda,sp)} = X - [F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)}
\end{aligned}$$

and hence $[F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$.

(5) $\Rightarrow$ (3): Let K_1, K_2 be any (Λ, sp)-closed sets of Y. By (4), we have

$$\begin{aligned}
&[[F^-([K_1]_{(\Lambda,sp)}) \cup F^+([K_2]_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\
&\subseteq [F^-([K_1]_{(\Lambda,sp)}) \cup F^+([K_2]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \\
&\subseteq F^-([[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \\
&\subseteq F^-(K_1^{(\Lambda,sp)}) \cup F^+(K_2^{(\Lambda,sp)}) = F^-(K_1) \cup F^+(K_2).
\end{aligned}$$

(3) $\Rightarrow$ (4): Let V_1, V_2 be any (Λ, sp)-open sets of Y. By (3), we have

$$\begin{aligned}
&X - [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)} \\
&= [F^-(Y - V_1^{(\Lambda,sp)}) \cup F^+(Y - V_2^{(\Lambda,sp)})]^{s(\Lambda,sp)} \\
&\subseteq F^-([Y - V_1^{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([Y - V_2^{(\Lambda,sp)}]^{(\Lambda,sp)}) \\
&= F^-(Y - [V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+(Y - [V_2^{(\Lambda,sp)}]_{(\Lambda,sp)}) \\
&\subseteq F^-(Y - V_1) \cup F^+(Y - V_2) = X - (F^+(V_1) \cap F^-(V_2))
\end{aligned}$$

and hence $F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)}$.

(4) $\Rightarrow$ (1): Let $x \in X$ and let V_1, V_2 be any (Λ, sp)-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By (4), we have

$$F^+(V_1) \cap F^-(V_2) \subseteq [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)}.$$

Put $U = [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{(\Lambda,sp)}$, then U is a $s(\Lambda, sp)$-open set of X containing x such that $F(U) \subseteq V_1^{(\Lambda,sp)}$ and $F(z) \cap V_2^{(\Lambda,sp)} \neq \emptyset$ for every $z \in U$. This shows that F is weakly quasi (Λ, sp)-continuous. □

Lemma 3.3. [2] *For a subset A of a topological space (X, τ), the following properties hold:*

(1) If A is (Λ, sp)-open in (X, τ), then $A^{(\Lambda,sp)} = A^{\theta(\Lambda,sp)}$.

(2) $A^{\theta(\Lambda,sp)}$ is (Λ, sp)-closed for every subset A of X.

Lemma 3.4. *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $A^{s(\Lambda,sp)} = A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(2) $A_{s(\Lambda,sp)} = A \cap [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

Theorem 3.5. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *F is weakly quasi* (Λ, sp)*-continuous;*

(2) $[F^-([B_1^{\theta(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([B_2^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(B_1^{\theta(\Lambda,sp)}) \cup F^+(B_2^{\theta(\Lambda,sp)})$ *for every subsets* B_1, B_2 *of* Y*;*

(3) $[F^-([B_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([B_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(B_1^{\theta(\Lambda,sp)}) \cup F^+(B_2^{\theta(\Lambda,sp)})$ *for every subsets* B_1, B_2 *of* Y*;*

(4) $[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* (Λ, sp)*-open sets* V_1, V_2 *of* Y*;*

(5) $[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* $p(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*;*

(6) $[F^-([K_1]_{(\Lambda,sp)}) \cup F^+([K_2]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(K_1) \cup F^+(K_2)$ *for every* $r(\Lambda, sp)$*-closed sets* K_1, K_2 *of* Y*.*

Proof. (1) ⇒ (2): Let B_1, B_2 be any subsets of Y. Then, $B_1^{\theta(\Lambda,sp)}$ and $B_2^{\theta(\Lambda,sp)}$ are (Λ, sp)-closed in Y. Thus, by Theorem 3.2 and Lemma 3.4,

$$\begin{aligned}
&[F^-([B_1^{\theta(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([B_2^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \\
&= [F^-([B_1^{\theta(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([B_2^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})] \\
&\cup [[F^-([B_1^{\theta(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([B_2^{\theta(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\
&\subseteq F^-(B_1^{\theta(\Lambda,sp)}) \cup F^+(B_2^{\theta(\Lambda,sp)}).
\end{aligned}$$

(2) ⇒ (3): This is obvious since $B^{(\Lambda,sp)} \subseteq B^{\theta(\Lambda,sp)}$ for every subset B of Y.

(3) ⇒ (4): This is obvious since $V^{(\Lambda,sp)} = V^{\theta(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y.

(4) ⇒ (5): Let V_1, V_2 be any $p(\Lambda, sp)$-open sets of Y. Then, since $V_i \subseteq [V_i^{(\Lambda,sp)}]_{(\Lambda,sp)}$, we have $V_i^{(\Lambda,sp)} = [[V_i^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for $i = 1, 2$. Now, put $G_i = [V_i^{(\Lambda,sp)}]_{(\Lambda,sp)}$, then G_i is (Λ, sp)-open in Y and $G_i^{(\Lambda,sp)} = V_i^{(\Lambda,sp)}$, by (4), $[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$.

(5) ⇒ (6): Let K_1, K_2 be any $r(\Lambda, sp)$-closed sets of Y. Then, $[K_1]_{(\Lambda,sp)}$ and $[K_2]_{(\Lambda,sp)}$ are $p(\Lambda, sp)$-open sets in Y, by (5),

$$\begin{aligned}
&[F^-([K_1]_{(\Lambda,sp)}) \cup F^+([K_2]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \\
&= [F^-([[[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([[[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \\
&\subseteq F^-([[K_1]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([[K_2]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \subseteq F^-(K_1) \cup F^+(V_2).
\end{aligned}$$

(6) $\Rightarrow$ (1): Let V_1, V_2 be any (Λ, sp)-open sets of Y. Then, $V_1^{(\Lambda,sp)}$ and $V_2^{(\Lambda,sp)}$ are $r(\Lambda, sp)$-closed sets of Y. Thus,

$$\begin{aligned}[F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)} &\subseteq [F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \\ &\subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)}).\end{aligned}$$

It follows from Theorem 3.2 that F is weakly quasi (Λ, sp)-continuous. $\square$

Theorem 3.6. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F *is weakly quasi* (Λ, sp)*-continuous;*

(2) $[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* $\beta(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*;*

(3) $[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* $s(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*;*

(4) $[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* $p(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $\beta(\Lambda, sp)$-open sets of Y. Then, $V_i \subseteq [[V_i^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and $V_i^{(\Lambda,sp)} = [[V_i^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for $i = 1, 2$. Since $V_1^{(\Lambda,sp)}$ and $V_2^{(\Lambda,sp)}$ are $r(\Lambda, sp)$-closed sets in Y, by Theorem 3.5, we have

$$[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)}).$$

(2) $\Rightarrow$ (3): This is obvious since $s\Lambda_{sp}O(Y, \sigma) \subseteq \beta\Lambda_{sp}O(Y, \sigma)$.

(3) $\Rightarrow$ (4): For any $V \in p\Lambda_{sp}O(Y, \sigma)$, $V^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed and $V^{(\Lambda,sp)} \in s\Lambda_{sp}O(Y, \sigma)$.

(4) $\Rightarrow$ (1): Let V_1, V_2 be any (Λ, sp)-open sets of Y. Then, V_1 and V_2 are $p(\Lambda, sp)$-open sets in Y, by (4),

$$\begin{aligned}[F^-(V_1) \cup F^+(V_2)]^{(\Lambda,sp)} &\subseteq [F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{s(\Lambda,sp)} \\ &\subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)}).\end{aligned}$$

It follows from Theorem 3.5 that F is weakly quasi (Λ, sp)-continuous. $\square$

Theorem 3.7. *For a multifunction* $F : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) *F is weakly quasi* (Λ, sp)*-continuous;*

(2) $[[F^-(V_1) \cup F^+(V_2)]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* $p(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*;*

(3) $[F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)} \subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)})$ *for every* $p(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*;*

(4) $F^-(V_1) \cup F^+(V_2) \subseteq [F^+(V_1^{(\Lambda,sp)}) \cup F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)}$ *for every* $p(\Lambda, sp)$*-open sets* V_1, V_2 *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V_1, V_2 be any $p(\Lambda, sp)$-open sets of Y. Since F is weakly quasi (Λ, sp)-continuous, by Theorem 3.5,

$$\begin{aligned}&[[F^-(V_1) \cup F^+(V_2)]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &\subseteq [[F^-([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cup F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &\subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)}).\end{aligned}$$

(2) $\Rightarrow$ (3): Let V_1, V_2 be any $p(\Lambda, sp)$-open sets of Y. By (2) and Lemma 3.4, we have

$$\begin{aligned}[F^-(V_1) \cup F^+(V_2)]^{s(\Lambda,sp)} &= [F^-(V_1) \cup F^+(V_2)] \cup [[F^-(V_1) \cup F^+(V_2)]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\ &\subseteq F^-(V_1^{(\Lambda,sp)}) \cup F^+(V_2^{(\Lambda,sp)}).\end{aligned}$$

(3) $\Rightarrow$ (4): Let V_1, V_2 be any $p(\Lambda, sp)$-open sets of Y. Thus, by (3),

$$\begin{aligned}&X - [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)}\\ &= [X - [F^+(V_1^{(\Lambda,sp)}) \cap F^-(V_2^{(\Lambda,sp)})]]^{s(\Lambda,sp)}\\ &= [[X - F^+(V_1^{(\Lambda,sp)})] \cup [X - F^-(V_2^{(\Lambda,sp)})]]^{s(\Lambda,sp)}\\ &= [F^-(Y - V_1^{(\Lambda,sp)}) \cup F^+(Y - V_2^{(\Lambda,sp)})]^{s(\Lambda,sp)}\\ &\subseteq F^-([Y - V_1^{(\Lambda,sp)}]^{(\Lambda,sp)}) \cup F^+([Y - V_2^{(\Lambda,sp)}]^{(\Lambda,sp)})\\ &= [X - F^+([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)})] \cup [X - F^+([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]\\ &\subseteq X - [F^+([V_1^{(\Lambda,sp)}]_{(\Lambda,sp)}) \cap F^-([V_2^{(\Lambda,sp)}]_{(\Lambda,sp)})]\\ &\subseteq X - [F^+(V_1) \cap F^-(V_2)]\end{aligned}$$

and hence $F^-(V_1) \cup F^+(V_2) \subseteq [F^+(V_1^{(\Lambda,sp)}) \cup F^-(V_2^{(\Lambda,sp)})]_{s(\Lambda,sp)}$.

(4) $\Rightarrow$ (1): Since every (Λ, sp)-open set is $p(\Lambda, sp)$-open, this follows from Theorem 3.2. □

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