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Applications of numerical radius inequalities

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Abstract

Let B, C be operators in the Hilbert-Schmidt class. Then

$$w_2\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + \frac{|w_2(B+C) - w_2(B-C)|}{\sqrt{2}} \leq \sqrt{2}\left(w_2(B) + w_2(C)\right).$$

Several Hilbert-Schmidt numerical radius inequalities are also given.

1 Introduction

Let $B(H)$ denote the set of all bounded linear operators on a complex separable Hilbert space H. For $A \in H$, let $\|A\|_p, \|A\|$, and let $\|A\|_2$ denote the Schatten $p-$norm, and in particular, the spectral norm and the Hilbert-Schmidt norm of A, respectively. Note that the Schatten $p-$norm for $1 \leq p < \infty$ of the operator $A \in B(H)$ is defined by $\|A\|_p = (tr\,|A|^p)^{1/p} = \sum_{j=1}^{\infty} s_j^p(A)$,

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where $s_j(A) = \lambda_j(|A|)$ is the singular value of the operator A. For recent studies about singular values, we refer the reader to [2-6] and [8-11]. The Schatten $p-$class, for $0 < p < \infty$, is denoted by C_p which consists of all operators A for which $\|A\|_p$ is finite. In particular, if $p = \infty$ we denote the norm by $\|A\|$ and is called spectral norm, and if $p = 2$ we denote the norm by $\|A\|_2$ and it is called Hilbert-Schmidt norm. It is known that for $A \in B(H)$, the numerical radius of A is given by

$$w(A) = \sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|.$$

A generalization of the numerical radius has been given in [12] by defining

$$w_N(A) = \sup_{\theta\in\mathbb{R}} N\left(\operatorname{Re}\left(e^{i\theta}A\right)\right),$$

for every $A \in B(H)$, where $N(.)$ is a norm in $B(H)$. In particular, if $N = \|.\|_2$, then $w_2(A) = \sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|_2$ is called Hilbert-Schmidt numerical radius. If A is self-adjoint, then $w_N(A) = N(A)$. It is known that the Hilbert Schmidt numerical radius and, in particular, the numerical radius define norms on $B(H)$.

For more details and proofs about numerical radius, we refer the reader to [1], [13], [14], [15], [16], and [18].

In [13], Al-Natoor and Audeh have provided a refinement of triangle inequality for the Schatten $p-$norm:

If $B, C \in C_p$, then

$$\|B + C\|_p \leq \sqrt{2}w_p \begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \leq \|B\|_p + \|C\|_p. \tag{1.1}$$

In particular, If $p = 2$, then

$$\|B + C\|_2 \leq \sqrt{2}w_2 \begin{bmatrix} 0 & B \\ C^* & 0 \end{bmatrix} \leq \|B\|_2 + \|C\|_2. \tag{1.2}$$

We provide several Hilbert-Schmidt numerical radius inequalities.

2 Main Results

We present the following two lemmas (the second lemma is well-known) which are essential in proving the next theorem .

Lemma 2.1. *Let* $A \in C_2$. *Then*

$$w_2 \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \sqrt{2}\, \|A\|_2 \,. \tag{2.3}$$

Proof.

$$\begin{aligned} w_2^2 \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} &= \frac{1}{2} \left\| \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \right\|_2^2 + \frac{1}{2} \left| tr \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}^2 \right| \\ &= \frac{1}{2} \left(\|A\|_2^2 + \|A^*\|_2^2 \right) + \frac{1}{2} \left| tr \begin{bmatrix} AA^* & 0 \\ 0 & A^*A \end{bmatrix} \right| \\ &= \|A\|_2^2 + \frac{1}{2} \left(tr\,(AA^*) + tr\,(A^*A) \right) \\ &= \|A\|_2^2 + \frac{1}{2} \left(\|A^*\|_2^2 + \|A\|_2^2 \right) = 2\, \|A\|_2^2 \,. \end{aligned}$$

This implies that

$$w_2 \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \sqrt{2}\, \|A\|_2 \,.$$

□

Lemma 2.2. *Let* s, t *be two real numbers. Then*

$$\frac{s+t}{2} = \max(s,t) - \frac{|s-t|}{2}. \tag{2.4}$$

We present here the following finding.

Theorem 2.3. *Let* $B, C \in C_2$. *Then*

$$w_2 \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + \frac{|w_2(B+C) - w_2(B-C)|}{\sqrt{2}} \le \sqrt{2}\left(w_2(B) + w_2(C)\right). \tag{2.5}$$

In particular,

$$\frac{1}{2}\, \|B\|_2 + \frac{|\|\text{Re}B\| - \|\text{Im}B\||}{2} \le w_2(B). \tag{2.6}$$

Proof. Throughout the proof of this theorem, let

$$S = \left[\max\left(w_2(B+C), w_2(B-C)\right) - T\right],$$

$$T = \frac{|w_2(B+C) - w_2(B-C)|}{2},$$

Now,

$$\begin{aligned} w_2\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} &\le \sqrt{2}\left(\frac{w_2(B+C)+w_2(B-C)}{2}\right), \\ &\quad \text{(by the inequality (??))} \\ &= \sqrt{2}S \\ &\quad \text{(by the equality (2.4)} \\ &\le \sqrt{2}\left[w_2(B)+w_2(C)-T\right]. \end{aligned}$$

This implies that

$$w_2\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + \frac{|w_2(B+C)-w_2(B-C)|}{\sqrt{2}} \le \sqrt{2}\left(w_2(B)+w_2(C)\right),$$

which is inequality (2.5).

To prove inequality (2.6), replace C with B^* in inequality (2.5) to get

$$w_2\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} + \frac{|w_2(B+B^*)-w_2(B-B^*)|}{\sqrt{2}} \le \sqrt{2}\left(w_2(B)+w_2(B^*)\right).$$

This gives

$$\frac{1}{\sqrt{2}}w_2\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} + \frac{|w_2(B+B^*)-w_2(B-B^*)|}{2} \le 2w_2(B).$$

Applying inequality (2.3) leads to

$$\|B\|_2 + |\|\mathrm{Re}B\|_2 - \|\mathrm{Im}B\|_2| \le 2w_2(B),$$

which implies that

$$\frac{1}{2}\|B\|_2 + \frac{|\|\mathrm{Re}B\|_2 - \|\mathrm{Im}B\|_2|}{2} \le w_2(B),$$

as required in the inequality (2.6). □

The next theorem is another application of the inequalities (2.3), (2.4).

Theorem 2.4. *Let* $B, C \in C_2$, $E = w_2\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, $F = \frac{\|B\|_2+\|C\|_2}{2}$,

$G=\frac{\left|\sqrt{2}w_2(B+C)-\frac{\|B\|_2+\|C\|_2}{2}\right|}{2}$, $I=\frac{\left|\sqrt{2}w_2(B-C)-\frac{\|B\|_2+\|C\|_2}{2}\right|}{2}$. Then

$$E+F+G+I\leq 2\left(w_2(B)+w_2(C)\right). \tag{2.7}$$

In particular,

$$\frac{\sqrt{2}+1}{4\sqrt{2}}\|B\|_2+\left|\frac{\|\mathrm{Re}B\|_2}{4}-\frac{\|B\|_2}{8\sqrt{2}}\right|+\left|\frac{\|\mathrm{Im}B_2\|_2}{4}-\frac{\|B\|_2}{8\sqrt{2}}\right|\leq w_2(B). \tag{2.8}$$

Proof. Throughout the proof of this theorem, let $E^*=w_2\begin{bmatrix}0 & B\\ B & 0\end{bmatrix}$, $F^*=\frac{\|B\|_2+\|B\|_2}{2}$,
$G^*=\frac{\left|\sqrt{2}w_2(B+B)-\frac{\|B\|_2+\|B\|_2}{2}\right|}{2}$, and $I^*=\frac{\left|\sqrt{2}w_2(B-B)-\frac{\|B\|_2+\|B\|_2}{2}\right|}{2}$,
Now,

$$\begin{aligned}
& w_2\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}+\frac{\|B\|_2+\|C\|_2}{2}\\
\leq\ & \frac{w_2(B+C)+w_2(B-C)}{\sqrt{2}}+\frac{\|B\|_2+\|C\|_2}{2}\\
=\ & \frac{\sqrt{2}w_2(B+C)+\frac{\|B\|_2+\|C\|_2}{2}}{2}+\frac{\sqrt{2}w_2(B-C)+\frac{\|B\|_2+\|C\|_2}{2}}{2}\\
=\ & \max\left(\sqrt{2}w_2(B+C),\frac{\|B\|_2+\|C\|_2}{2}\right)-G\\
& +\max\left(\sqrt{2}w_2(B-C),\frac{\|B\|_2+\|C\|_2}{2}\right)-I\\
\leq\ & 2\sqrt{2}\left(w_2(B)+w_2(C)\right)-G-I.
\end{aligned}$$

This implies that,

$$E+F+G+I\leq 2\sqrt{2}\left(w_2(B)+w_2(C)\right).$$

This proves the inequality (2.7).

To prove the inequality (2.8), replace C by B^* in the inequality (2.7), we get:

$$E^*+F^*+G^*+I^*\leq 2\sqrt{2}\left(w_2(B)+w_2(B^*)\right).$$

This gives,

$$\begin{aligned} & (\sqrt{2}+1)\left\|B\right\|_2+\left|\sqrt{2}\left\|\mathrm{Re}B\right\|_2-\frac{\left\|B\right\|_2}{2}\right| \\ & +\left|\sqrt{2}\left\|\mathrm{Im}B\right\|_2-\frac{\left\|B\right\|_2}{2}\right| \\ \leq\ & 4\sqrt{2}w_2(B). \end{aligned}$$

Dividing both sides by $4\sqrt{2}$ yields

$$\frac{\sqrt{2}+1}{4\sqrt{2}}\left\|B\right\|_2+\left|\frac{\left\|\mathrm{Re}B\right\|_2}{4}-\frac{\left\|B\right\|_2}{8\sqrt{2}}\right|+\left|\frac{\left\|\mathrm{Im}B_2\right\|_2}{4}-\frac{\left\|B\right\|_2}{8\sqrt{2}}\right|\leq w_2(B),$$

which is the inequality (2.8). □

To prove the final result in this study, we need the following well known lemma.

Lemma 2.5. *Let B, C be positive operators in $B(H)$, $p \geq 1$. Then*

$$tr(B+C)^p \leq 2^{p-1}tr(B^p+C^p). \tag{2.9}$$

Theorem 2.6. *Let $A \in B(H)$ be positive operator, $p \geq 1$. Then*

$$w_p(A) \leq \sqrt[p]{\frac{1}{2}\left[tr(A)^p+tr(A^*)^p\right]}. \tag{2.10}$$

Proof.

$$\begin{aligned} \left\|\mathrm{Re}e^{i\theta}A\right\|_p^p &= tr\left(\mathrm{Re}e^{i\theta}A\right)^p \\ &= tr\left(\frac{e^{i\theta}A+e^{-i\theta}A^*}{2}\right)^p \\ &= \frac{1}{2^p}tr\left(e^{i\theta}A+e^{-i\theta}A^*\right)^p \\ &\leq \frac{1}{2^p}\times 2^{p-1}\left[tr\left(e^{i\theta}A\right)^p+tr\left(e^{-i\theta}A^*\right)^p\right] \\ &\quad \text{(by using inequalit (2.9))} \\ &= \frac{1}{2}\left[tr\left(e^{i\theta}A\right)^p+tr\left(e^{-i\theta}A^*\right)^p\right]. \end{aligned}$$

Now,

$$\begin{aligned} w_p^p(A) &= \sup_{\theta\in\mathbb{R}} \left\| \mathrm{Re} e^{i\theta} A \right\|_p^p \\ &= \sup_{\theta\in\mathbb{R}} \frac{1}{2}\left[tr\left(e^{i\theta}A\right)^p + tr\left(e^{-i\theta}A^*\right)^p \right] \\ &\leq \frac{1}{2}\left[tr(A)^p + tr(A^*)^p \right]. \end{aligned}$$

This implies that,

$$w_p(A) \leq \sqrt[p]{\frac{1}{2}\left[tr(A)^p + tr(A^*)^p \right]},$$

which is inequality (2.10). □

Remark 2.7. *In the special case, when $p = 1$, we have*

$$w_1(A) \leq Re\left(tr(A)\right)$$

References

[1] W. Audeh, M. Al-Labadi, Some results about numerical radius inequalities, Int. J. Math. Comput. Sci., **17,** no. 1, (2022), 33–39.

[2] W. Audeh, Singular value inequalities with applications, J. Math. Computer Sci., **24,** (2022), 323–329.

[3] W. Audeh, Singular value inequalities for operators and matrices, Ann. Funct. Anal., (2022).

[4] W. Audeh, Singular value inequalities and applications, Positivity,**25,** (2021), 843–852.

[5] W. Audeh, Singular value and norm inequalities of Davidson-Power type, J. Math. Inequal., **15,** (2021), 1311–1320.

[6] W. Audeh, Some generalizations for singular value inequalities of compact operators, Adv. Oper. Theory, **6,** (2021).

[7] W. Audeh, Hilbert-Schmidt numerical radius inequalities for 2*2 operator matrices, Int. J. Math. Comput. Sci., **16,** no. 4, (2021), 1161–1167.

[8] W. Audeh, Norm inequalities and applications, Int. J. Math. Comput. Sci., **16,** no. 2, (2021), 809–817.

[9] W. Audeh, Generalizations for singular value and arithmetic-geometric mean inequalities of operators, J. Math. Anal. Appl., **489,** no. 2, (2020), 124184.

[10] W. Audeh, Generalizations for singular value inequalities of operators, Adv. Oper. Theory, **5,** (2020), 371–381.

[11] W. Audeh, F. Kittaneh, Singular value inequalities for compact operators, Linear Algebra Appl., **437,** (2012), 2516–2522.

[12] A. Abu-Omar, F. Kittaneh, A generalization of the numerical radius, Linear Algebra Appl., **569,** (2019), 323–334.

[13] A. Al-Natoor, W. Audeh, Refinement of triangle inequality using numerical radius, Adv. Oper. Theory, **5,** (2020), 1635–1645.

[14] R. Bhatia, Matrix Analysis, Springer, New York, 1997.

[15] K. E. Gustafson, D. K. M. Rao, Numerical range, Springer, New York, 1997.

[16] P. R. Halmos, A Hilbert space problem book, 2nd ed. Springer, New York, 1982.

[17] O. Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices, Integr. Equ. Oper. Theory, **71,** (2011), 129–147.

[18] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, Studia Math., **178,** (2007), 83–89.