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The Transmuted Power Hazard Rate Distribution and its Applications

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Abstract

In this work, we present a transmuted power hazard rate distribution. We discuss its structural properties. The estimation of parameters are acquired via maximum likelihood method. The application indicates that the proposed model fits data better compare to the classical model.

1 Introduction

In last three decades, there has been a growing trend in transmuted distributions and many of them have been reported in literature. This model received more consideration after the work of Shaw and Buckely (2009). The extended form of the probability distribution provides major applicability in many area.

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There has been a trend among the researchers to modify/develop new techniques for generalizing new models of distributions. An extension of the classical distribution are more capable for modeling data in areas such as finance, economics, engineering, demography, reliability analysis and medical sciences. [2, 3, 10, 11, 14, 6, 7, 15, 17]
A random variable X has the power hazard rate distribution (PHRD), if its probability density function (pdf) is given as

$$g(x) = \gamma x^{\delta} \exp\left\{-\frac{\gamma}{\delta+1} x^{\delta+1}\right\}, \quad x > 0, \tag{1.1}$$

and its cumulative distribution function (cdf) as

$$G(x) = 1 - \exp\left\{-\frac{\gamma}{\delta+1} x^{\delta+1}\right\}, \quad x > 0, \tag{1.2}$$

where $\gamma > 0$ and $\delta > -1$.
For more details on PHRD (1.1), we can refer [5, 12, 13, 18, 19] among others. The remainder of this article is presented as follows. The main distribution is presented in Section 2. Some of its properties are introduced in Section 3. Parameters' estimation are planned in Section 4. Some applications are provided in Section 5. Section 6 addresses the concluding remarks.

2 Transmuted Power Hazard Rate Distribution

The transmuted power hazard rate distribution (TPHRD) is proposed in this section.

Definition 2.1. *If X has cdf, $G(x)$, the transmuted distribution of X has the following cdf*

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \quad |\lambda| \leq 1. \tag{2.3}$$

From (1.2) and (2.3), the cdf of the TPHRD with parameters γ, δ and λ can be presented as follows

$$F(x) = \left(1 - e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\left[1 + \lambda - \lambda\left(1 - e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\right], \quad x > 0, \tag{2.4}$$

The pdf of TPHRD in view of (2.4) is

$$f(x) = \gamma x^{\delta} e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\left[1 - \lambda + 2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right], \quad x > 0. \tag{2.5}$$

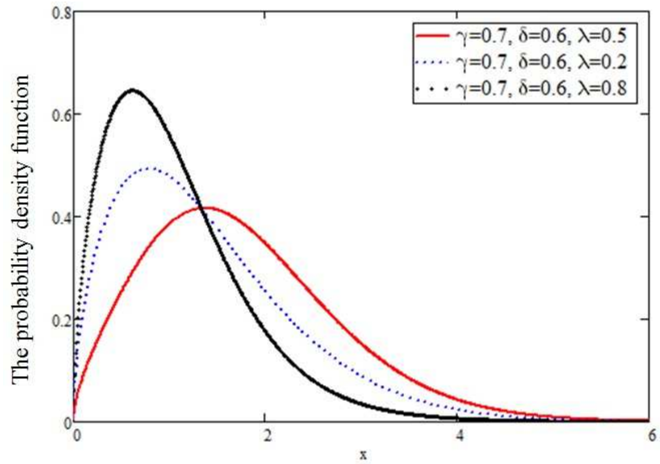

Figure 1: The pdf of the TPHRD(0.7,0.6,λ).

When $\lambda = 0$, we have base distribution. The pdf plots of the TPHRD$(0.7, 0.6, \lambda)$ are shown in Figure 1, for different values of λ.
From Figure 1, the TPHRD has unimode.

2.1 Statistical properties

Some statistical properties of the TPHRD are addressed below.

Theorem 2.2. *If $X \sim$ TPHRD$(\gamma, \delta, \lambda)$, then the rth moment of X is given as follows*

$$\mu_r^{'} = \left[1 - \lambda\left(1 - 2^{-\frac{r}{\delta+1}}\right)\right]\left(\frac{\delta+1}{\gamma}\right)^{\frac{r}{\delta+1}}\Gamma\left(1 + \frac{r}{\delta+1}\right). \tag{2.6}$$

Proof.
The rth moment is defined as

$$\mu_r^{'} = E[X^r] = \int_0^\infty x^r f(x)dx,$$

from (2.5), then

$$\begin{aligned}
\mu_r^{'} &= \gamma\int_0^\infty x^{\delta+r}e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\left[1 - \lambda + 2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right]dx \\
&= \gamma(1-\lambda)\int_0^\infty x^{\delta+r}e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}dx + 2\gamma\lambda\int_0^\infty x^{\delta+r}e^{-\frac{2\gamma}{\delta+1}x^{\delta+1}}dx \\
&= I_1 + I_2. \qquad (2.7)
\end{aligned}$$

Let $u_1 = \frac{\gamma}{\delta+1}x^{\delta+1}$, then $x = \left(\frac{\delta+1}{\gamma}\right)^{\frac{1}{\delta+1}} u_1^{\frac{1}{\delta+1}}$ and $dx = \left(\frac{\delta+1}{\gamma}\right)^{\frac{1}{\delta+1}} \frac{1}{\delta+1} u_1^{\frac{1}{\delta+1}-1} du_1$. Upon simplification,

$$I_1 = (1-\lambda)\left(\frac{\delta+1}{\gamma}\right)^{\frac{r}{\delta+1}} \Gamma\left(1+\frac{r}{\delta+1}\right). \tag{2.8}$$

Also, let $u_2 = \frac{2\gamma}{\delta+1}x^{\delta+1}$, then $x = \left(\frac{\delta+1}{2\gamma}\right)^{\frac{1}{\delta+1}} u_2^{\frac{1}{\delta+1}}$ and $dx = \left(\frac{\delta+1}{2\gamma}\right)^{\frac{1}{\delta+1}} \frac{1}{\delta+1} u_2^{\frac{1}{\delta+1}-1} du_2$. Upon simplification,

$$I_2 = 2^{-\frac{r}{\delta+1}}\lambda\left(\frac{\delta+1}{\gamma}\right)^{\frac{r}{\delta+1}} \Gamma\left(1+\frac{r}{\delta+1}\right). \tag{2.9}$$

Substituting from (2.8) and (2.9) into (2.7), we have

$$\mu_r^{'} = \left[1-\lambda+\lambda\left(2^{-\frac{r}{\delta+1}}\right)\right]\left(\frac{\delta+1}{\gamma}\right)^{\frac{r}{\delta+1}} \Gamma\left(1+\frac{r}{\delta+1}\right). \tag{2.10}$$

This completes the proof.
At $r = 1$, in (2.6), $E(X)$ can be obtained as follows

$$\mu_1^{'} = \left[1-\lambda+\lambda\left(2^{-\frac{1}{\delta+1}}\right)\right]\left(\frac{\delta+1}{\gamma}\right)^{\frac{1}{\delta+1}} \Gamma\left(1+\frac{1}{\delta+1}\right). \tag{2.11}$$

If $r = 2$, in (2.6), $E(X^2)$ as follows

$$\mu_2^{'} = \left[1-\lambda+\lambda\left(2^{-\frac{2}{\delta+1}}\right)\right]\left(\frac{\delta+1}{\gamma}\right)^{\frac{2}{\delta+1}} \Gamma\left(1+\frac{2}{\delta+1}\right). \tag{2.12}$$

The $Var(X)$ of TPHRD is given by

$$\begin{aligned} Var(X) &= \mu_2^{'} - \mu_1^{'2} \\ &= \left(\frac{\delta+1}{\gamma}\right)^{\frac{2}{\delta+1}}\left\{\left[1-\lambda+\lambda\left(2^{-\frac{2}{\delta+1}}\right)\right]\Gamma\left(1+\frac{2}{\delta+1}\right)\right. \\ &\quad \left. -\left[1-\lambda+\lambda\left(2^{-\frac{1}{\delta+1}}\right)\right]^2\Gamma^2\left(1+\frac{1}{\delta+1}\right)\right\}. \end{aligned} \tag{2.13}$$

From (2.6), the skewness (S_k) and kurtosis (K_u) can be estimated by practicing the below relation.

$$S_k = \frac{\mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2\mu_1^{'3}}{(\mu_2^{'} - \mu_1^{'})^{3/2}}, \qquad K_u = \frac{\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6\mu_1^{'2}\mu_2^{'} - 3\mu_1^{'4}}{(\mu_2^{'} - \mu_1^{'})^2}.$$

Theorem 2.3. *If $X \sim TPHRD(\gamma, \delta, \lambda)$, then the moment generating function (MGF) of X is given as follows*

$$M_X(t) = \sum_{r=0}^{\infty} \left[1 - \lambda\left(1 - 2^{-\frac{r}{\delta+1}}\right)\right] \left(\frac{\delta+1}{\gamma}\right)^{\frac{r}{\delta+1}} \Gamma\left(1 + \frac{r}{\delta+1}\right) \frac{t^r}{r!}. \quad (2.14)$$

Proof.
We have

$$M_X(t) = E[e^{Xt}] = \int_0^{\infty} e^{xt} f(x) dx,$$

but $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$, then

$$M_X(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(xt)^r}{r!} f(x) dx = \sum_{r=0}^{\infty} E(X^r) \frac{t^r}{r!}, \quad (2.15)$$

to make use of (2.6), Theorem 2.3 is completes.

2.2 Quantiles and Random Number generation

The pth quantiles can be obtained by using

$$F(x_p; \gamma, \delta, \lambda) = p. \quad (2.16)$$

x_p satisfies the following equation, by using (2.4) and (2.16).

$$\left(1 - e^{-\frac{\gamma}{\delta+1} x_p^{\delta+1}}\right) \left[1 + \lambda - \lambda\left(1 - e^{-\frac{\gamma}{\delta+1} x_p^{\delta+1}}\right)\right] - p = 0,$$

let $u = e^{-\frac{\gamma}{\delta+1} x_p^{\delta+1}}$, then

$$\lambda u^2 + (1-\lambda)u + (p-1) = 0. \quad (2.17)$$

Therefore, the pth percentile can be calculated as follows

$$x_p = \left\{ -\left(\frac{\delta+1}{\gamma}\right) \ln\left[\frac{-(1-\lambda) + \sqrt{(1-\lambda)^2 - 4\lambda(p-1)}}{2\lambda}\right] \right\}^{\frac{1}{\delta+1}}. \quad (2.18)$$

Equation (2.18) gives the median, m, by setting $p = 0.5$.

$$m = \left\{ -\left(\frac{\delta+1}{\gamma}\right) \ln\left[\frac{-(1-\lambda) + \sqrt{1+\lambda^2}}{2\lambda}\right] \right\}^{\frac{1}{\delta+1}}. \quad (2.19)$$

The random numbers can be generated from the TPHRD as

$$F(x;\gamma,\delta,\lambda)=u,$$

where u has a uniform distribution in $(0,1)$. After simplification, we have

$$x=\left\{-\left(\frac{\delta+1}{\gamma}\right)\ln\left[\frac{-(1-\lambda)+\sqrt{(1-\lambda)^2-4\lambda(u-1)}}{2\lambda}\right]\right\}^{\frac{1}{\delta+1}}. \quad (2.20)$$

Given the values of γ,δ and λ, above equation can be used to generate random numbers.

2.3 Reliability analysis

The reliability (survival) function of TPHRD is given as

$$R(x)=\bar{F}(x)=\lambda e^{-\frac{2\gamma}{\delta+1}x^{\delta+1}}+(1-\lambda)e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}, \quad x>0. \quad (2.21)$$

The hazard rate (HR) of TPHRD takes the form

$$h(x)=\frac{f(x)}{R(x)}=\gamma x^{\delta}\left[\frac{1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}}{1-\lambda+\lambda e^{-\frac{2\gamma}{\delta+1}x^{\delta+1}}}\right], \quad x>0. \quad (2.22)$$

The reversed hazard (RH) rate function

$$r(t)=\frac{f(x)}{F(x)}=\frac{\gamma x^{\delta}e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\left[1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right]}{\left(1-e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\left(1+\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)}. \quad (2.23)$$

The Mean residual life (MRL) of TPHRD is

$$\begin{aligned}\mathrm{MRL}(t) &= \int_0^{\infty} R(x|t)dx \\ &= \frac{1}{(\delta+1)R(t)}\left(\frac{\delta+1}{\gamma}\right)^{\frac{1}{\delta+1}}\left[\lambda 2^{-\frac{1}{\delta+1}}\gamma\left(\frac{2\gamma}{\delta+1}x^{\delta+1},\frac{1}{\delta+1}\right)\right. \\ &\left. +(1-\lambda)\gamma\left(\frac{\gamma}{\delta+1}x^{\delta+1},\frac{1}{\delta+1}\right)\right], \quad (2.24)\end{aligned}$$

where $\gamma(a,b)=\int_a^{\infty}u^{b-1}e^{-u}du$, is an incomplete gamma function.

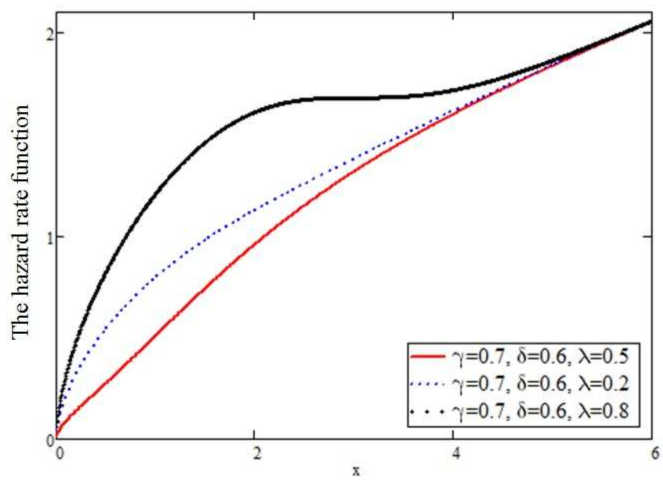

Figure 2: The HRF of the TPHRD(0.7, 0.6,λ).

Figure 2, depicts the some hazard rate plots of the TPHRD with indicated parameter values.
From Figure 2, the HRF of the TPHRD is increasing and unimodal.

Table 1 contains the values of some statistical measures for TPHRD(0.7, 0.6,λ) for $\lambda \in (-1, 1)$. For the PHRD, we have $\mu_X = 1.503$, Med=2.257, Var(X) $= 0.925$, sk=0.962, ku=4.044 and $cv = 63.991$, for the same values of parameters.

Table 1: Some statistical measures for TPHRD(0.7,0.6,λ) for $\lambda \in (-1, 1)$.

δ	$E(X)$	Median	$Var(X)$	S_k	K_u	CV
-1	2.031	1.906	0.903	0.774	3.779	46.79
-0.8	1.926	1.805	0.952	0.735	3.691	50.66
-0.6	1.82	1.695	0.979	0.749	3.663	54.37
-0.4	1.714	1.577	0.983	0.798	3.706	57.85
-0.2	1.609	1.455	0.965	0.871	3.831	61.05
0	1.503	1.333	0.925	0.962	4.044	63.99
0.2	1.397	1.217	0.863	1.065	4.356	66.50
0.4	1.292	1.11	0.778	1.172	4.77	68.27
0.6	1.186	1.016	0.67	1.263	5.247	69.02
0.8	1.08	0.934	0.541	1.277	5.531	68.10
1	0.975	0.864	0.389	0.962	4.044	63.97

From Table 1, we observe that:

1. the TPHRD is positive skewed, for all values of λ, also PHRD is positive skewed.
2. when $0 < \lambda < 1$, the TPHRD and PHRD are highly skewed, $(S_k > 1)$.
3. the dispersion for the distributions are increasing for λ increasing.
4. the TPHRD and PHRD are leptokurtic $(S_k > 3)$ for all λ.
5. Since the coefficient of variation $(Cv = \frac{\sqrt{Var(X)}}{mean} \times 100)$ is larger for PHRD for $\lambda < 0$, the TPHRD are more variable than the PHRD, for all values of $\lambda > 0$.

Therefore, the TPHR model is more flexible than PHR model.

2.4 Order statistics

Suppose $X_1, X_2, \cdots, X_m$ be a random sample of size m, taking from a distribution with cdf $F(x)$ and pdf $f(x)$. The $X_{(1)} < X_{(2)} < \cdots < X_{(m)}$ are the corresponding order statistics (O.S.). The pdf of $X_{(\ell)}$ is given by

$$f_{X(\ell)}(x) = \frac{m!}{(\ell-1)!(m-\ell)!} [F(x)]^{\ell-1} [1-F(x)]^{m-\ell} f(x), \quad \text{for } \ell = 1, 2, \cdots, m, \tag{2.25}$$

substituting from (2.4) and (2.5) into (2.25), the pdf of ℓth order statistics for TPHRD$(\gamma, \delta, \lambda)$ is given by

$$\begin{aligned} f_{X(\ell)}(x) &= \frac{m!}{(\ell-1)!(m-\ell)!}\gamma x^{\delta} \left[1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right] e^{-(m-\ell+1)\frac{\gamma}{\delta+j}x^{\delta+1}} \times \\ & \left[\left(1-e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\left(1+\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\right]^{\ell-1} \left[(1-\lambda)+\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right]^{m-\ell}. \end{aligned} \tag{2.26}$$

From (2.26), the pdf of largest and smallest O.S. $X_{(m)}(X_{(1)})$ are given sa follows

$$\begin{aligned} f_{X(m)}(x) &= m\gamma x^{\delta} e^{-\frac{\gamma}{\delta+1}x^{\delta+1}} \left[1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right] \times \\ & \left[\left(1-e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\left(1+\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right)\right]^{m-1}, \end{aligned} \tag{2.27}$$

$$f_{X(1)}(x) = m\gamma x^{\delta} e^{-\frac{m\gamma}{\delta+1}x^{\delta+1}} \left[1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right] \left[(1-\lambda)+\lambda e^{-\frac{\gamma}{\delta+1}x^{\delta+1}}\right]^{m-1}. \tag{2.28}$$

3 Estimation

The parameters of TPHRD can be estimated by using the maximum likelihood estimation (MLE). Suppose $X_1, X_2, \cdots, X_m$ be a random sample from a population having TPHRD$(\gamma, \delta, \lambda)$. The likelihood function is

$$L(\gamma,\delta,\lambda;x)=\prod_{i=1}^{m} f(x_i;\gamma,\delta,\lambda)=\prod_{i=1}^{m}\gamma x_i^{\delta} e^{-\frac{\gamma}{\delta+1}x_i^{\delta+1}}\left(1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right), \quad x>0. \tag{3.29}$$

The $\mathcal{L}=\ln L(\gamma,\delta,\lambda;x)$ is

$$\mathcal{L}=m\ln(\gamma)+\delta\sum_{i=1}^{m}\ln(x_i)-\frac{\gamma}{\delta+1}\sum_{i=1}^{m}x_i^{\delta+1}+\ln\left(1-\lambda+2\lambda e^{-\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right). \tag{3.30}$$

Equation (3.30) can be differentiated with respect to γ, δ and λ, and equating the derivatives to zero, the normal equations are obtained.

$$\frac{\partial\mathcal{L}}{\partial\gamma} = \frac{m}{\gamma}-\frac{1}{\delta+1}\sum_{i=1}^{m}x_i^{\delta+1}-\frac{2\lambda}{\delta+1}\sum_{i=1}^{m}\frac{x_i^{\delta+1}}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}. \tag{3.31}$$

$$\frac{\partial\mathcal{L}}{\partial\delta} = \sum_{i=1}^{m}\ln(x_i)+\frac{\gamma}{(\delta+1)^2}\sum_{i=1}^{m}x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right]+\frac{2\lambda\gamma}{(\delta+1)^2}\times \sum_{i=1}^{m}\frac{x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right]}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}=0, \tag{3.32}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \sum_{i=1}^{m}\frac{2-e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}=0. \tag{3.33}$$

Since the MLEs cannot be derived analytically, we do not have their actual distributions that can allow us to derive the exact confidence intervals of the parameters. Alternatively, we can derive the asymptotic confidence intervals using the asymptotic behavior of the MLEs. It is well known that the MLE of $\Lambda=(\gamma,\delta,\lambda)$, say $\hat{\Lambda}$, follows approximately a multivariate normal distribution with mean of Λ and variance-covariance matrix equals the inverse of the observed Fisher information matrix, see [8]. That is, $\hat{\Lambda}=\left(\hat{\gamma},\hat{\delta},\hat{\lambda}\right)\sim \mathbf{N_3}(\mathbf{\Lambda},\mathbf{V})$, where $\mathbf{V}$ is given as follows

$$\mathbf{V} = \begin{bmatrix} -\frac{\partial^2\mathcal{L}}{\partial\gamma^2} & -\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\delta} & -\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\lambda} \\ -\frac{\partial^2\mathcal{L}}{\partial\delta\partial\gamma} & -\frac{\partial^2\mathcal{L}}{\partial\delta^2} & -\frac{\partial^2\mathcal{L}}{\partial\delta\partial\lambda} \\ -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\gamma} & -\frac{\partial^2\mathcal{L}}{\partial\lambda\partial\delta} & -\frac{\partial^2\mathcal{L}}{\partial\lambda^2} \end{bmatrix}^{-\mathbf{1}} = \begin{bmatrix} var(\hat{\gamma}) & cov(\hat{\gamma},\hat{\delta}) & cov(\hat{\gamma},\hat{\lambda}) \\ cov(\hat{\delta},\hat{\gamma}) & var(\hat{\delta}) & cov(\hat{\delta},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\gamma}) & cov(\hat{\lambda},\hat{\delta}) & var(\hat{\lambda}) \end{bmatrix}. \tag{3.34}$$

where

$$\frac{\partial^2\mathcal{L}}{\partial\gamma^2} = -\frac{m}{\gamma^2} + \frac{2\lambda(1-\lambda)}{(\delta+1)^2}\sum_{i=1}^{m}\frac{x_i^{2(\delta+1)}e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}{\left[2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right]^2}, \tag{3.35}$$

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\delta} &= \frac{1}{(\delta+1)^2}\sum_{i=1}^{m}x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right] + \frac{2\lambda}{(\delta+1)^2}\sum_{i=1}^{m}\frac{x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right]}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}} \\ &\quad -\frac{2\lambda(1-\lambda)\gamma}{(\delta+1)^3}\sum_{i=1}^{m}\frac{x_i^{2(\delta+1)}e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\left[1-(\delta+1)\ln(x_i)\right]}{\left[2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right]^2},\end{aligned} \tag{3.36}$$

$$\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\lambda} = -\frac{2}{\delta+1}\sum_{i=1}^{m}\frac{x_i^{\delta+1}}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}} + \frac{2\lambda}{\delta+1}\sum_{i=1}^{m}\frac{x_i^{\delta+1}\left[2-e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right]}{\left[2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right]^2}, \tag{3.37}$$

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial\delta^2} &= -\frac{2\gamma}{(\delta+1)^3}\sum_{i=1}^{m}x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right] \\ &\quad +\frac{\gamma}{(\delta+1)^2}\sum_{i=1}^{m}x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right]\ln(x_i) \\ &\quad -\frac{2\lambda\gamma}{(\delta+1)^3}\sum_{i=1}^{m}\frac{x_i^{\delta+1}\left[2-2(\delta+1)\ln(x_i)+(\delta+1)^2(\ln(x_i))^2\right]}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}} \\ &\quad +\frac{2\lambda(1-\lambda)\gamma^2}{(\delta+1)^4}\sum_{i=1}^{m}\frac{x_i^{2(\delta+1)}\left[1-(\delta+1)\ln(x_i)\right]^2e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}{\left[2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right]^2},\end{aligned} \tag{3.38}$$

$$\frac{\partial^2\mathcal{L}}{\partial\delta\partial\lambda} = \frac{2\gamma}{(\delta+1)^2}\sum_{i=1}^{m}\frac{x_i^{\delta+1}\left[1-(\delta+1)\ln(x_i)\right]e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}{\left[2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}\right]^2}, \tag{3.39}$$

$$\frac{\partial^2\mathcal{L}}{\partial\lambda^2} = -\sum_{i=1}^{m}\left[\frac{2-e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}{2\lambda+(1-\lambda)e^{\frac{\gamma}{\delta+1}x_i^{\delta+1}}}\right]^2. \tag{3.40}$$

A $100(1-\alpha)\%$ confidence interval for Λ_j, $j=1,2,3$, can be approximated by $\hat{\Lambda}_j \pm z_{\frac{\alpha}{2}}\, SE_j\left(\hat{\Lambda}_j\right)$, where $z_{\frac{\alpha}{2}}$ is the upper $100\frac{\alpha}{2}$-th percentile of the standard normal distribution and $SE_j\left(\hat{\Lambda}_j\right)$ is the square root of the $j-$th element in the diagonal of $\mathbf{V}$.

4 Applications

Two real data sets are analyzed. The comparing between TPHRD and PHRD are presented in this section. Some criteria are used to compare TPHRD and PHRD as follows.

1. The AIC (Akaikes information criterion): [1]

$$\text{AIC} = 2k - 2\mathcal{L},$$

2. The AICC (corrected Akaikes information criterion): [4]

$$\text{AAIC} = \text{AIC} + \frac{2k(k+1)}{m-k+1}.$$

3. The BIC (Bayesian information criterion): [16]

$$\text{BIC} = k\ln(m) - 2\mathcal{L},$$

   where k and m are the number of parameters and observed data, respectively.

4. The R^2 (coefficient of determination):

$$R^2 = \frac{\sum_{i=1}^{m}\left(\hat{F}(x_i)-\bar{F}\right)^2}{\sum_{i=1}^{m}\left(\hat{F}(x_i)-\bar{F}\right)^2 + \sum_{i=1}^{m}\left(F_n(x_i)-\hat{F}(x_i)\right)^2},$$

   where $\hat{F}(x)$ is estimated CDF,

$$\bar{F}(x) = \frac{1}{m}\sum_{i=1}^{m}\hat{F}(x_i),$$

$F_m(x)$ is the empirical distribution function,

$$F_m(x) = \frac{1}{m}\sum_{i=1}^{m} I(x_{(i)} \leq x),$$

and

$$I(x_{(i)} \leq x) = \begin{cases} 1 & \text{if} \quad x_{(i)} \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

5. The RMSE (root mean square error):

$$RMSE = \left[\frac{1}{m}\sum_{i=1}^{m}\left(F_m(x_i) - \hat{F}(x_i)\right)^2\right]^{\frac{1}{2}}.$$

6. The $K - S$ (KolmogorovSmirnov statistic) for a given CDF $\hat{F}(x)$ is

$$\text{K-S} = \sup_x |F_m(x) - \hat{F}(x)|.$$

If the data have a larger value of R^2 and smaller values of AIC, AAIC, BIC, $K - S$ and RMSE, it indicates that proposed model can be taken as a best fit.

Example 4.1: The below mentioned data is an uncensored data set comprising of 100 readings on breaking stress of carbon fibers (in Gba), [9].

0.920	0.928	0.997	0.9971	1.061	1.117	1.162	1.183	1.187	1.192
1.196	1.213	1.215	1.2199	1.220	1.224	1.225	1.228	1.237	1.240
1.244	1.259	1.261	1.263	1.276	1.310	1.321	1.329	1.331	1.337
1.351	1.359	1.388	1.408	1.449	1.4497	1.450	1.459	1.471	1.475
1.477	1.480	1.489	1.501	1.507	1.515	1.530	1.5304	1.533	1.544
1.5443	1.552	1.556	1.562	1.566	1.585	1.586	1.599	1.602	1.614
1.616	1.617	1.628	1.684	1.711	1.718	1.733	1.738	1.743	1.759
1.777	1.794	1.799	1.806	1.814	1.816	1.828	1.830	1.884	1.892
1.944	1.972	1.984	1.987	2.020	2.0304	2.029	2.035	2.037	2.043
2.046	2.059	2.111	2.165	2.686	2.778	2.972	3.504	3.863	5.306

The parameters' estimation, K–S and corresponding p-value for PHRD and TPHRD are reported in Table 2. Table 3 reports, the values of $\mathcal{L}$, AIC, AAIC, BIC, RMSE and R^2.

Table 2: MLEs of γ, δ, λ and KolmogorovSmirnov statistic.

Model	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\lambda}$	K–S	p-value
PHRD	0.521	1.632	–	0.195135	0.000823
TPHRD	0.285	1.952	0.84	0.181729	0.002316

Table 3: The $\mathcal{L}$, AIC, AIC, BIC, RMSE and R^2.

Model	$\mathcal{L}$	AIC	AAIC	BIC	RMSE	R^2
PHRD	-90.149	184.298	184.422	189.509	0.109576	0.768124
TPHRD	-83.222	172.443	172.693	180.259	0.008983	0.832837

On comparing both Tables 2 and 3, we come across TPHRD gives better fit to the data over PHRD.
The variance-covariance matrix is given as

$$\mathbf{V} = \begin{bmatrix} 1.671 \times 10^{-3} & -4.111 \times 10^{-3} & -1.705 \times 10^{-3} \\ -4.111 \times 10^{-3} & 0.032 & 4.587 \times 10^{-4} \\ -1.705 \times 10^{-3} & 4.587 \times 10^{-4} & 0.011 \end{bmatrix}.$$

Then the 95% C.I. for γ, δ and λ of TPHRD as $(0.20518, 0.36544)$, $(1.59955, 2.30401)$ and $(0.63681, 1)$, respectively.
The likelihood function has unique solution, as depicted in Figure 3.

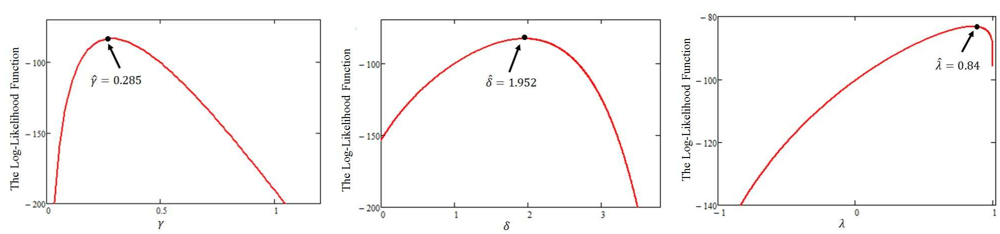

Figure 3: The depiction of $\mathcal{L}$ of γ, δ and λ.

For $\hat{\gamma} = 0285$, $\hat{\delta} = 1.952$ and $\hat{\lambda} = 0.84$, some statistical measures can be calculated, see Table 4.

Table 4: Some statistical measures, for TPHRD at $\hat{\gamma}, \hat{\delta}$ and $\hat{\lambda}$.

Mean	Median	Variance	Skewness	Kurtosis
1.623	1.594	0.384	0.335	3.047

Notice from Table 4:

1. the TPHRD is right skewed ($S_k > 0$) and it approximately symmetric skewed ($-0.5 < S_k < 0.5$).
2. the TPHRD is platykurtic. ($K_u < 3$).

Example 4.2: The following data set is generated data to simulate the strengths of glass fibers, [9].

1.014	1.081	1.082	1.185	1.223	1.248	1.267	1.271	1.272	1.275
1.276	1.278	1.286	1.288	1.292	1.304	1.306	1.355	1.361	1.364
1.379	1.409	1.426	1.459	1.46	1.476	1.481	1.484	1.501	1.506
1.524	1.526	1.535	1.541	1.568	1.579	1.581	1.591	1.593	1.602
1.666	1.670	1.684	1.691	1.704	1.731	1.735	1.747	1.748	1.757
1.800	1.806	1.867	1.876	1.878	1.910	1.916	1.972	2.012	2.456
2.592	3.197	4.121							

Table 5 contains the values of MLEs of γ, δ, λ, K–S and the corresponding p-value for PHRD and TPHRD. The values of $\mathcal{L}$, AIC, AAIC, BIC, RMSE and R^2 are computed in Table 6.

Table 5: The MLEs of γ, δ, λ and KolmogorovSmirnov statistic.

Model	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\lambda}$	K–S	p-value
PHRD	0.517	2.062	–	0.205069	0.008434
TPHRD	0.276	2.451	0.851	0.191863	0.016675

Table 6: The $\mathcal{L}$, AIC, AIC, BIC, RMSE and R^2.

Model	$\mathcal{L}$	AIC	AAIC	BIC	RMSE	R^2
PHRD	-46.367	96.734	96.934	101.02	0.119557	0.723197
TPHRD	-41.728	89.457	89.863	95.886	0.010837	0.795499

As per comparison from Tables 5–6, we arrive at, the proposed model could work better than PHRD model.
The variance-covariance matrix is given as

$$\mathbf{V} = \begin{bmatrix} 2.599 \times 10^{-3} & -8.415 \times 10^{-3} & -1.968 \times 10^{-3} \\ -8.415 \times 10^{-3} & 0.07 & 7.707 \times 10^{-4} \\ -1.968 \times 10^{-3} & 7.707 \times 10^{-4} & 0.013 \end{bmatrix}.$$

Then the 95% C.I. of γ, δ and λ for TPHRD as $(0.17628, 0.37611)$, $(1.93205, 2.96896)$ and $(0.62656, 1)$, respectively.

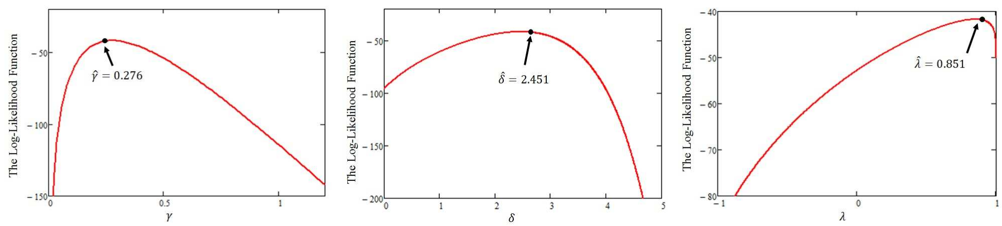

Figure 4: The shape of $\mathcal{L}$ of γ, δ and λ.

Figure 4 shows that the solution of likelihood function is unique.
For $\hat{\gamma} = 0.276$, $\hat{\delta} = 2.451$ and $\hat{\lambda} = 0.851$, some statistical measures can be calculated, see Table 7.

Table 7: Some statistical measures, for TPHRD at $\hat{\gamma}, \hat{\delta}$ and $\hat{\lambda}$.

Mean	Median	Variance	Skewness	Kurtosis
1.579	1.57	0.272	0.168	2.92

From Table 7, we observe that:

1. The TPHRD is right skewed ($S_k > 0$) and it approximately symmetric skewed ($-0.5 < S_k < 0.5$).

2. The TPHRD is platykurtic. ($K_u < 3$).

5 Conclusion

In this article, we proposed the transmuted power hazard rate distribution via a transmutation approach. The analytical and practical interpretation are studied in broadly. We hope that TPHRD may be applied in many different areas due to its better performance over the PHRD as seen some applications section.

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