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Near Ring Multiplications on a Locally Compact Topological Ring

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Abstract

In this paper, we show ways to define near ring products on R^n, where $(R, +, \cdot)$ is a locally compact topological ring and R^n is equipped with coordinatewise addition.

1 Introduction

Magill, Jr. [2], [3] considered the problem of defining near products on the Euclidean groups $(\mathbb{R}^n, +)$. His pioneering work showed the close interplay between the algebraic and topological properties of $\mathbb{R}^n$, especially successful when $n = 1$. Other interesting studies on defining near multiplications appeared in [1,5,6].

Definition 1.1. [2] *Let* $(S, +)$ *and* $(S, *)$ *be topological groupoids. The triple* $(S, +, *)$ *is a*

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1. **right distributive topological system** *(denoted by Rdts) if* $(a+b)*c = a*c + b*c$ *for all* $a, b, c \in S$;

2. **left distributive topological system** *(denoted by Ldts) if* $a*(b+c) = a*b + a*c$ *for all* $a, b, c \in S$;

3. **doubly distributive topological system** *(hereafter, denoted by Ddts) if it is both an Ldts and an Rdts.*

Definition 1.2. [4] *An algebraic system* $N = (N, +, \cdot)$ *is called a* **right near ring** *if it satisfies the following conditions:*

1. $(N, +)$ *is a group (not necessarily abelian);*

2. $(N, \cdot)$ *is a semigroup;*

3. $(x+y)\cdot z = x\cdot z + y\cdot z$ *for all* x, y, z *in* N.

Theorem 1.3. [3] *Let* $(S, +)$ *be a locally compact Hausdorff groupoid and let* t *be a continuous map from* $(S, +)$ *into* $End(S)$. *Let* $t(y) = f_y$ *for each* $y \in S$ *and define* $x * y = f_y(x)$ *for all* $x, y \in S$. *Then* $(S, +, *)$ *is an Rdts and every multiplication* $*$ *such that* $(S, +, *)$ *is an Rdts is obtained in this manner. Furthermore, the operation* $*$ *is associative so that* $(S, +, *)$ *is a semigroup if and only if* $f_x o f_y = f_{f_x(y)}$ *for all* $x, y \in S$.

Corollary 1.4. [3] *Let* $(S, +)$ *be a group and, in addition, let* S *be a locally compact Hausdorff space where the binary operation* '$+$' *is continuous. Let* t *be a continuous map from* S *into* $End(S)$ *where* $t(x) = f_x$. *Define a binary operation* '$*$' *on* S *by* $x * y = f_y(x)$. *Then* $(S, +, *)$ *is an Rdts and every binary operation* $*$ *for which* $(S, +, *)$ *is an Rdts is obtained in this manner. Furthermore, the operation* $*$ *is associative and hence* $(S, +, *)$ *is a topological near ring if and only if* $f_x o f_y = f_{f_x(y)}$ *for all* $x, y \in S$.

Theorem 1.5. [3] *Let* $(S, +)$ *be a locally compact Hausdorff groupoid and* $Con(S)$ *the space of all continuous selfmaps of* S. *Let* t *be a continuous homomorphism from* $(S, +)$ *into* $(Con(S), \oplus)$. *Let* $t(y) = f_y$ *for each* $y \in S$ *and define* $x * y = f_y(x)$ *for all* $x, y \in S$. *Then* $(S, +, *)$ *is an Ldts and every multiplication* $*$ *such that* $(S, +, *)$ *is an Ldts is obtained in this manner. Again, the operation* $*$ *is associative so that* $(S, +, *)$ *is a semigroup if and only if* $f_x o f_y = f_{f_x(y)}$ *for all* $x, y \in S$.

Corollary 1.6. [3] *Let $(S, +)$ be a abelian group and, in addition let S be a locally compact Hausdorff space where the binary operation $+$ is continuous. Let t be a continuous homomorphism from $(S, +)$ into $(End(S), \oplus)$ where $t(x) = f_x$. Define a binary operation $*$ on S by $x * y = f_y(x)$. Then $(S, +, *)$ is a Ddts and every binary operation $*$ for which $(S, +, *)$ is a Ddts is obtained in this manner. Furthermore the binary operation $*$ is associative and hence $(S, +, *)$ is a topological ring if and only if $f_x o f_y = f_{f_x(y)}$ for all $x, y \in S$.*

Theorem 1.7. *(Theorem* 3.1 *of* [3]*) Let f be a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$ and define $v * w = f(w)v$. Then $(\mathbb{R}^n, +, *)$ is a topological near ring if and only if $f(av) = af(v)$ for all $v \in R^n$ and all $a \in Ran(f)$ where $Ran(f)$ is the range of a function f. It is a topological ring if and only if f is a linear map.*

2 Main Results

We prove the following generalization of Lemma 2.9 of [3] for arbitrary locally compact topological rings. Lemma 2.9 of [3] was proved for Euclidean rings $\mathbb{R}^n$.
Let $(R, +, \cdot)$ be a locally compact topological ring. Let k be a positive integer. Choose k^2 continuous mappings $f_{ij}, 1 \leq i, j \leq k$ and for each w in R^k, let A_w be the matrix $[f_{ij}(w)]$ and $End(R^k)$ be the class of all continuous homomorphisms on $(R^k, +, \cdot)$ and equip R^k with the compact open topology.

Lemma 2.1. *If $(R, +)$ is abelian, then the map t from R^k into $End(R^k)$ defined by $t(w) = [f_{ij}(w)]$ is linear and continuous. Every linear continuous map from R^k into $End(R^k)$ is obtained in this manner. Furthermore, t is a homomorphism from $(R^k, +, \cdot)$ into $(End(R^k), \oplus, \odot)$ if and only if each f_{ij} is a linear map from $(R^k, \oplus, \odot)$ into $(R, +, \cdot)$.*

Proof. We show first that t is continuous.
Let $v \in R^k, K \subset R^k, G \subset R^k$, K be compact and G be open.
Suppose $t(v) \in [K, G]$ where $[K, G]$ denotes the set of all continuous functions $\phi : R^k \to R^k$ which satisfy $\phi(K) \subseteq G$. Then $t(v)(x) = [f_{ij}(v)](x) \in G$.
Define a function F from $R^k \times R^k$ into R^k by
$F(w, y) = [f_{ij}(w)](y)$ (y is a column vector). Then F is continuous since each f_{ij} is continuous.
Since $F(v, x) \in G$ and G is open, there exist neighbourhoods V_x and W_x of v and x respectively such that $F(V_x \times W_x) \subseteq G$.
Then $\{W_x\}_{x \in K}$ is an open cover of K and since K is compact, there exists a

finite subcover $\{W_{x_i}\}_{i=1}^{m}$.
Let $V = \bigcap_{i=1}^{m} V_{x_i}$. Choose any $w \in V$ and any $y \in K$.
Then $y \in W_{x_i}$ for some i and we have $(t(w))(y) = [f_{ij}(w)](y) = F(w, y)$ $\in F[V_{x_i} \times W_{x_i}] \subseteq G$.
Thus $t(w) \in [K, G]$ for all $w \in V$ and since V is a neighbourhood of v, we conclude that t is continuous.
Now suppose that t is a continuous linear function from R^k into $End(R^k)$.
Then for each $v \in R^k$, $t(v) : (R^k, +, \cdot) \to (R^k, +, \cdot)$ is linear: $t(v)(w_1 + w_2) = t(v)w_1 + t(v)w_2$ and $t(v)(\alpha w) = \alpha t(v)w$ for all $w_1, w_2, w \in R^k$ and $\alpha \in R$.
We will identify $t(v)$ with the matrix that induces that map. Since the entries of the matrix depend on the vector v, it follows that there exist k^2 functions $f_{ij}, 1 \leq i, j \leq k$ such that $t(v) = [f_{ij}(v)]$.
It remains to show that each f_{ij} is continuous.
Let e_j denote the element of R^k having 1 in the j^{th} place and $0's$ elsewhere.
Define $E_j : EndR^k \to R^k$ by $E_j(f) = A_f e_j$ where A_f is a $k \times k$ matrix associated with f.
Finally, let π_i denote the i^{th} projection map from R^k to R. Clearly E_j is continuous.
Now$(\pi_i \circ E_j ot)(v) = \pi_i \circ (E_j(t(v)))$
$(\pi_i \circ E_j ot)(v) = \pi_i(t(v)(e_j)) = \pi_i([f_{ij}(v)](e_j)) = f_{ij}(v)$.
The continuity of π_i, E_j and t ensures that of f_{ij}.

$$\begin{aligned}
&\text{Finally } t(x+y) = t(x) + t(y) \\
&\quad \text{if and only if } [f_{ij}(x+y)] = [f_{ij}(x)] + [f_{ij}(y)] \\
&\quad \text{if and only if } f_{ij}(x+y) = f_{ij}(x) + f_{ij}(y) \text{ for each } f_{ij} \text{ and} \\
&\qquad\qquad t(\lambda x) = \lambda t(x) \\
&\quad \text{if and only if } [f_{ij}](\lambda x) = [\lambda f_{ij}(x)] \\
&\quad \text{if and only if } f_{ij}(\lambda x) = \lambda f_{ij}(x).
\end{aligned}$$

This completes the proof of the ***Lemma*2.1**. As an immediate consequence of ***Corollary*** [**1.4**] and ***Lemma*** [**2.1**], we have

Theorem 2.2. *Suppose R has multiplicative identity 1. Define $v * w = A_w(v)$. Then $(R^k, +, *)$ is an Rdts and every Rdts whose additive group is $(R^k, +)$ can be obtained in this manner. Furthermore the operation $*$ is associative and hence $(R^k, +, *)$ is a locally compact topological near ring if and only if $A_v A_w = A_{A_v(w)}$ for all $v, w \in R^k$.*

As an immediate consequence of ***Corollary*** [**1.6**] and ***Lemma*** [**2.1**] we have the following

Theorem 2.3. *Define $v * w = A_w(v)$. Then $(R^k, +, *)$ is a Ddts and every Ddts whose additive group $(R^k, +)$ is obtained in exactly this manner. Furthermore, the operation $*$ is associative and hence $(R^k, +, *)$ is a topological ring if and only if $A_v A_w = A_{A_v(w)}$ for all $v, w \in R^k$.*

Remark 2.4. *If the hypothesis of **Lemma** [2.1] is strengthened by assuming $(R, +, \cdot)$ to be the real number field $(\mathbb{R}, +, \cdot)$ then we can weaken the hypothesis in another direction and obtain the same conclusion as the following corollary shows.*

Corollary 2.5. *Choose n^2 continuous functions $f_{ij}, 1 \leq i, j \leq n$ from $\mathbb{R}^n$ into $\mathbb{R}$ and for each $w \in \mathbb{R}^n$, let A_w be the matrix $[f_{ij}(w)]$. Then the map t from $\mathbb{R}^n$ into End$(\mathbb{R}^n)$ defined by $t(w) = [f_{ij}(w)]$ is continuous and every continuous map from $\mathbb{R}^n$ into End$(\mathbb{R}^n)$ is obtained in this manner. Furthermore, t is a homomorphism from $(\mathbb{R}^n, +)$ into (End$(\mathbb{R}^n, \oplus)$ if and only if each f_{ij} is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.*

Proof. It remains to show $t(v)$ is not just additive but also homogeneous;i.e. $t(v)(\alpha w) = \alpha t(v)(w)$ for all $\alpha \in \mathbb{R}$ and $w \in \mathbb{R}^k$.
$t(v)(1w) = t(v)(w) = 1t(v)w$. If $(t(v)(nw) = nt(v)w$ for a positive integer n, then

$$t(v)((n+1)w) = t(v)(nw+w) = t(v)(nw)+t(v)(w) = nt(v)(w)+t(v)w = (n+1)t(v)(w)$$

. By induction $t(v)(\lambda w) = \lambda t(v)w$ for all $\lambda \in \mathbb{N}$.
It is immediate that $t(v)(\lambda w) = \lambda t(v)w$ for all $\lambda \in \mathbb{Z}$. $t(v)(w) = t(v)(n\frac{1}{n}w) = nt(v)(\frac{1}{n}w)$
$\Rightarrow t(v)(\frac{1}{n}w) = \frac{1}{n}t(v)(w)$. It is immediate that $t(v)(\lambda w) = \lambda t(v)w$ for all λ in $\mathbb{Q}$. If λ is a real number then let $\{\lambda_n\}$ be a rational sequence converging to λ. Then
$t(v)(\lambda w) = \lim t(v)(\lambda_n w) = \lim \lambda_n t(v)(w) = \lambda t(v)w$ proving that t is also homogeneous and hence linear.

3 Conclusion

In this paper, we defined near ring products on R^n, where $(R, +, \cdot)$ is a locally compact topological ring and R^n is equipped with coordinate wise addition. Ideal structures and other interesting properties of near rings can be considered for further study.

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