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(Λ, sp)-continuity and $\delta(\Lambda, sp)$-closed sets

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Abstract

In this paper, we characterize upper and lower (Λ, sp)-continuous multifunctions.

1 Introduction

Topology is concerned with all questions directly or indirectly related to continuity. Semi-open sets, pre-open sets, α-open sets, β-open sets and δ-open sets play important roles in the generalizations of continuity in topological spaces. Using these sets, many authors introduced and studied various types of weak forms of continuity for functions and multifunctions. In 1993, Popa and Noiri [9] obtained several characterizations of upper and lower α-continuous multifunctions. Popa and Noiri [8] introduced and investigated the concepts of upper and lower β-continuous multifunctions. In 2000, Noiri and Popa [5] investigated the further characterizations of upper and lower M-continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. In 2004, Park et al. [6] introduced and studied δ-precontinuous multifunctions as a generalization of precontinuous multifunctions due to Popa [7]. In [3], the author introduced the concepts

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of upper and lower (Λ, sp)-continuous multifunctions. The purpose of the present paper is to characterize upper and lower (Λ, sp)-continuous multifunctions by utilizing $\delta(\Lambda, sp)$-closed sets.

2 Preliminaries

Throughout this paper and unless explicitly stated, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset A of a topological space (X, τ) is called *β-open* [1] if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. A subset $\Lambda_{sp}(A)$ [4] is defined as follows:

$$\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}.$$

If $A = \Lambda_{sp}(A)$, then A is called a *Λ_{sp}-set* [4]. A subset A of a topological space (X, τ) is said to be *(Λ, sp)-closed* [3] if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called *(Λ, sp)-open*. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, sp)-cluster point* [3] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the *(Λ, sp)-closure* [3] of A and is denoted by $A^{(\Lambda, sp)}$. The union of all (Λ, sp)-open sets contained in A is called the *(Λ, sp)-interior* [3] of A and is denoted by $A_{(\Lambda, sp)}$. A subset A of a topological space (X, τ) is said to be *$s(\Lambda, sp)$-open* if $A \subseteq [A_{(\Lambda, sp)}]^{(\Lambda, sp)}$ [3]. The complement of a $s(\Lambda, sp)$-open set is said to be *$s(\Lambda, sp)$-closed*. The family of all $s(\Lambda, sp)$-open sets in a topological space (X, τ) is denoted by $s\Lambda_{sp}O(X, \tau)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. Following [2], or a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively; that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 (Λ, sp)-continuity and $\delta(\Lambda, sp)$-closed sets

In this section, we characterize upper and lower (Λ, sp)-continuous multifunctions.

Definition 3.1. [3] *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) upper (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \subseteq V$, there exists a (Λ, sp)-open set U of X containing x such that $F(U) \subseteq V$;

(ii) lower (Λ, sp)-continuous if, for each $x \in X$ and each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a (Λ, sp)-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.

Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a $\delta(\Lambda, sp)$-*cluster point* of A if $A \cap [U^{(\Lambda,sp)}]_{(\Lambda,sp)} \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all $\delta(\Lambda, sp)$-cluster points of A is called $\delta(\Lambda, sp)$-*closure* of A and is denoted by $A^{\delta(\Lambda,sp)}$. If $A = A^{\delta(\Lambda,sp)}$, then A is said to be $\delta(\Lambda, sp)$-*closed*. The complement of a $\delta(\Lambda, sp)$-closed set is said to be $\delta(\Lambda, sp)$-*open*. The union of all $\delta(\Lambda, sp)$-open sets contained in A is called the $\delta(\Lambda, sp)$-*interior* of A and is denoted by $A_{\delta(\Lambda,sp)}$.

Lemma 3.2. *Let A be a subset of a topological space (X, τ). Then, the following properties hold:*

(1) If A is (Λ, sp)-open in X, then $A^{(\Lambda,sp)} = A^{\delta(\Lambda,sp)}$.

(2) $A^{\delta(\Lambda,sp)}$ is (Λ, sp)-closed.

Definition 3.3. *A topological space (X, τ) is called $s(\Lambda, sp)$-regular if, for each $s(\Lambda, sp)$-closed set F and each $x \notin F$, there exist disjoint $s(\Lambda, sp)$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 3.4. *A topological space (X, τ) is $s(\Lambda, sp)$-regular if and only if for each $x \in X$ and each $s(\Lambda, sp)$-open set U containing x, there exists a $s(\Lambda, sp)$-open set V such that $x \in V \subseteq V^{s(\Lambda,sp)} \subseteq U$.*

Lemma 3.5. *Let (X, τ) be a $s(\Lambda, sp)$-regular space. Then, the following properties hold:*

(1) $A^{(\Lambda,sp)} = A^{\delta(\Lambda,sp)}$ for every subset A of X.

(2) Every (Λ, sp)-open set is $\delta(\Lambda, sp)$-open.

Theorem 3.6. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $s(\Lambda, sp)$-regular space, the following properties are equivalent:*

(1) F is upper (Λ, sp)-continuous;

(2) $F^-[B^{\delta(\Lambda,sp)}]$ is (Λ, sp)-closed in X for every subset B of Y;

(3) $F^-(K)$ is (Λ, sp)-closed in X for every $\delta(\Lambda, sp)$-closed set K of Y;

(4) $F^+(V)$ is (Λ, sp)-open in X for every $\delta(\Lambda, sp)$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 3.2, $B^{\delta(\Lambda,sp)}$ is (Λ, sp)-closed in Y and by Theorem 38 of [3], $F^-(B^{\delta(\Lambda,sp)})$ is (Λ, sp)-closed in X.

(2) $\Rightarrow$ (3): Let K be any $\delta(\Lambda, sp)$-closed set of Y. Then, $K^{\delta(\Lambda,sp)} = K$. By (2), we have $F^-(K) = F^-(K^{\delta(\Lambda,sp)})$ is (Λ, sp)-closed in X.

(3) $\Rightarrow$ (4): The proof is obvious.

(4) $\Rightarrow$ (1): Let V be any (Λ, sp)-open set of Y. Since (Y, σ) is $s(\Lambda, sp)$-regular, by Lemma 3.5, V is $\delta(\Lambda, sp)$-open in Y and by (4), $F^+(V)$ is (Λ, sp)-open in X. Thus, by Theorem 38 of [3], F is upper (Λ, sp)-continuous. □

Theorem 3.7. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $s(\Lambda, sp)$-regular space, the following properties are equivalent:*

(1) F is lower (Λ, sp)-continuous;

(2) $F^+[B^{\delta(\Lambda,sp)}]$ is (Λ, sp)-closed in X for every subset B of Y;

(3) $F^+(K)$ is (Λ, sp)-closed in X for every $\delta(\Lambda, sp)$-closed set K of Y;

(4) $F^-(V)$ is (Λ, sp)-open in X for every $\delta(\Lambda, sp)$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.6. □

Definition 3.8. [10] *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (Λ, sp)-continuous at a point $x \in X$ if, for each (Λ, sp)-open set V of Y containing $f(x)$, there exists a (Λ, sp)-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (Λ, sp)-continuous if f has this property at each point $x \in X$.*

Corollary 3.9. *For a function $f : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $s(\Lambda, sp)$-regular space, the following properties are equivalent:*

(1) f is (Λ, sp)-continuous;

(2) $f^{-1}[B^{\delta(\Lambda,sp)}]$ *is* (Λ, sp)*-closed in* X *for every subset* B *of* Y*;*

(3) $f^{-1}(K)$ *is* (Λ, sp)*-closed in* X *for every* $\delta(\Lambda, sp)$*-closed set* K *of* Y*;*

(4) $f^{-1}(V)$ *is* (Λ, sp)*-open in* X *for every* $\delta(\Lambda, sp)$*-open set* V *of* Y*.*

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