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On a Certain Subclass of Univalent Functions Involving the Beta Function

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Abstract

Investigation of coefficient problems is of great interest among function theorists, The nature of coefficients of series functions go along way to determine their properties. In this work therefore, the authors introduce a new subclass of univalent functions denoted by $A_{\varpi}(\alpha)$ and establish its coefficient estimate. Furthermore, class $A_{\varpi}(\alpha)$ involving beta function $B(m,n)$ is discussed.

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1 Introduction and Preliminaries

Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n > 0 \tag{1.1}$$

denoted by T is a subclass of A.

Porwal in [6] introduced univalent functions associated with Poisson distribution series in geometric function theory defined as

$$P_m(z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}e^{-m}}{(n-1)!} z^n. \tag{1.2}$$

This form of functions was also used by several authors such as See [1] and [7].

Univalent Function Involving Beta Function

Let $m, n > 0$ such that m and n are distinct values. That is, $m \neq n$. Then, we have that

$$f_{m,n}(z) = z - \sum_{n=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^n \tag{1.3}$$

is a normalized univalent function which belongs to class S with Beta function as its coefficients. Oluwayemi *et al.* in [2] introduced univalent function $F_{m,n}(z)$ associated with beta function $B(m,n)$ while Vanithaa *et al.* in [7] introduced function $L_m f(z)$ as a convolution of power series with the coefficients as Poisson distribution and the normalised univalent function of the form (1.1). Motivated by the works of [2] and [7], we introduce another form of univalent functions with the aid of Hadamard product. By convolution of (1.2) and (1.3), we define a function

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(m-1)!(n-1)!m^{n-1}e^{-m}}{(n-1)!(m+n-1)!} z^n = f_{m,n}(z) * P_m(z) \tag{1.4}$$

Let $(m-1)! = (n-1)! = 1$. That is, $m = n = 1$ in (1.3). Then, equation (1.4) above, reduces to (1.2) introduced by Porwal in [6]. That is, $f_{1,1}(z) * P_m(z) \equiv P_m(z)$. See [1], [3] [4], [5], and [7].

Definition 1.1. *Let $0 \leq \varpi \leq 1$ and $0 < \alpha \leq \frac{\pi}{2}$. A function $f(z) \in T$ defined by (1.1) is said to belong to a subclass $A_{\varpi}(\alpha)$ if it satisfies the following inequality condition:*

$$\Re\left\{\frac{zf'(z) + z^2 f''(z)}{\varpi zf'(z) + (1-\varpi)f(z)}\right\} > cos\alpha. \tag{1.5}$$

2 Main Results

2.1 Coefficient Estimate

Theorem 2.1. *Suppose the function $f(z)$ is defined by (1.1). Then, $f \in A_{\varpi}(\alpha)$ if*

$$\sum_{n=2}^{\infty} \{n[\varpi(n-1) + 1 - \varpi cos\alpha] + (1-\varpi)cos\alpha\}\, a_n \leq 1 - cos\alpha. \tag{2.6}$$

Proof: Suppose the inequality (2.6) is true and $|z| = 1$. Let $f \in A_{\varpi}(\alpha)$. From definition 1.1, we have

$$Re\left\{\frac{zf'(z) + z^2 f''(z)}{\varpi zf'(z) + (1-\varpi)f(z)}\right\} = Re\left\{\frac{z - \sum_{n=2}^{\infty} n[(n-1)+1]a_n z^n}{z - \sum_{n=2}^{\infty}[\varpi(n-1)+1]a_n z^n}\right\} \geq cos\alpha$$

$$= Re\left\{\frac{1 - \sum_{n=2}^{\infty} n[(n-1)+1]a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}[\varpi(n-1)+1]a_n z^{n-1}}\right\} \geq cos\alpha.$$

Taking the value of z on the real axis for $|z| = r \to 1^{-1}$, we have that

$$= Re\left\{\frac{1 - \sum_{n=2}^{\infty} n[(n-1)+1]a_n}{1 - \sum_{n=2}^{\infty}[\varpi(n-1)+1]a_n}\right\} \geq \eta.$$

Thus (2.6) holds true. Conversely, assume (2.6) holds true,

$$\left|\frac{zf'(z) + z^2 f''(z)}{\varpi zf'(z) + (1-\varpi)f(z)}\right| = \left|\frac{z - \sum_{n=2}^{\infty} n[(n-1)+1]a_n z^n}{z - \sum_{n=2}^{\infty}[\varpi(n-1)+1]a_n z^n}\right|$$

$$\leq \frac{cos\alpha\,(1 - \sum_{n=2}^{\infty} n[(n-1)+1]a_n)}{1 - \sum_{n=2}^{\infty}[\varpi(n-1)+1]a_n} \leq cos\alpha.$$

By the maximum modulus theorem, we have that $f \in A_{\varpi}(\alpha)$.

Corollary 2.2. *If $f \in A_{\varpi}(\alpha)$, then*

$$|a_n| \leq \frac{1 - cos\alpha}{n[(n-1) + 1 - \varpi cos\alpha] + (1-\varpi)cos\alpha}.$$

Equality holds for equation

$$f(z) = z - \frac{1 - cos\alpha}{n[(n-1) + 1 - \varpi cos\alpha] + (1-\varpi)cos\alpha}.$$

Theorem 2.3. *Suppose the function $f(z)$ defined by (1.4). Then, $f \in A_{\varpi}(\alpha)$ if*

$$\sum_{n=2}^{\infty} \left[m(1 - \varpi cos\alpha)(2 - e^{-m}) + \mu(m,n)\right] B() \leq 1 - \cos\alpha$$

where

$$\mu(m,n) = \frac{(-1)^n n m^{n-2} m^{n-1}}{(n-2)!}.$$

Proof:

Let $fz() \in A_{\varpi}(\alpha)$. Then from Theorem 2.1, it follows that

$$\sum_{n=2}^{\infty} \{n[(n-1) + 1 - \varpi cos\alpha] + (1-\varpi)cos\alpha\}\, a_n \leq 1 - cos\alpha.$$

Now using (1.4), we have

$$\sum_{n=2}^{\infty} \left[(n-1)(1 - \varpi cos\alpha + n)\frac{m^{n-1}e^{-m}}{(n-1)!} + (1 - cos\alpha)\frac{m^{n-1}e^{-m}}{(n-1)!}\right] \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right)$$

$$(\leq 1 - cos\alpha)$$

$$= e^{-m}\left\{(1 - \varpi cos\alpha)m\sum_{n=0}^{\infty}\frac{m^n}{n!} + (1 - cos\alpha)\sum_{n=1}^{\infty}\frac{m^n}{n!} + \sum_{n=2}^{\infty}\frac{n m^{n-1}}{(n-2)!}\right\}\left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right)$$

$$= e^{-m}\left\{(1 - \varpi cos\alpha)m e^m + (1 - cos\alpha)(e^m - 1) + \sum_{n=2}^{\infty}\frac{n m^{n-1}}{(n-2)!}\right\}\left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right)$$

$$= \left\{m(1 - \varpi cos\alpha)(2 - e^{-m}) + \sum_{n=2}^{\infty}\frac{(-1)^n m^{n-2}}{(n-2)!}.\sum_{n=2}^{\infty}\frac{n m^{n-1}}{(n-2)!}\right\}\left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right)$$

$$= \sum_{n=2}^{\infty} m(1-\varpi cos\alpha)(2-e^{-m}) \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) + \sum_{n=2}^{\infty} \frac{(-1)^n n m^{n-2} m^{n-1}}{(n-2)!} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right).$$

Since this is bounded by $1 - cos\alpha$, the result holds true.

Remark 2.4.

$$If\ e^m = 1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+\ldots = \sum_{n=1}^{\infty} \frac{m^n}{n!},\ \ then\ \ e^{-m} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} m^n}{n!} \equiv \sum_{n=2}^{\infty} \frac{(-1)^n m^{n-2}}{(n-2)!}.$$

References

[1] A. Altunhan, S. S. Eker, On certain subclasses of unvalent functions of complex order associated with poisson distribution series, *Boletin de la Sociedad Matematica Mexicana*, (2020) 1-12

[2] M. O. Oluwayemi, S. Olatunji, T. O. Ogunlade, On certain properties of univalent functions associated with Beta function, *Abstract and Applied Analysis*, (2022), (to appear).

[3] M. O. Oluwayemi, I. Faisal, A new family of analytic functions associated with multiplier transformation, *Scientific Africa*, **12**,(2021), 1-9 https://doi.org/10.1016/j.sciaf.2021.e00754.

[4] M. O. Oluwayemi, K. Vijaya, On a study of univalent functions associated with a multiplier transformation,*International Journal of Mathematics and Computer Science,* **17,** no. 2,(2022), 931–935.

[5] M. O. Oluwayemi, Alb Lupas, Cătaş, Results on a class of analytic functions related with multiplier transformation, *Scientific African*, **15**, (2022), e01115.

[6] S. Porwal, An application of a Poisson distribution series on certain analytic functions, *Journal of Complex Analysis*, **2014**, (2014), 1–3, doi: 10 1155/2014/984135.

[7] L. Vanitha, C. Ramachandran, T. Bulboaca, Certain subclasses of spirallike univalent functions related to Poission distribution series, *Turkish Journal of Mathematics*, **45**, (2021), 1449–1458, doi:10.3906/mat-2102-96.