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Analytical and Numerical Solutions to the Kapitza Pendulum Equation

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Abstract

We solve the Kapitza pendulum equation analytically and numerically. In the latter solution, we use the fourth-order Runge-Kutta (RK4) numerical method. The proposed methodology is beneficial in the study of nonlinear phenomena and control theory.

1 Introduction

Due to the potential applications of the simple nonlinear pendulum model, we study the inverted pendulum, also called the Kapitza pendulum. The

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inverted pendulum is a classic problem in dynamics and control theory and is used as a benchmark for testing control strategies. The differential equation that describes the behavior of the system is investigated and its analytical solution is obtained which allows for a better analysis of the problem.

2 Equation of motion for Kapitza pendulum

In this section, we construct the differential equation that describes the behavior of an inverted pendulum. Also, this equation of motion is called the Kapitza pendulum equation, after the Russian physicist Pyotr Kapitza who first analysed it [1]. We consider a simple, nonlinear pendulum of mass m and length l moving on a vertical plane in the uniform gravitational field and subjected to a vertical, rapid vibration of the pivot point. By rapid vibration, we mean oscillation of high frequency and small amplitude of the pivot motion (See Figure 1).

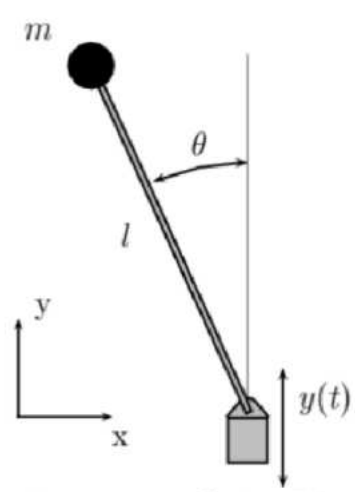

Figure 1. The diagram of the Kapitza pendulum.

The Lagrangian of the inverted pendulum with a vertically driven pivot reads

$$L = K - U = \frac{m}{2}(l^2\dot{\theta}^2 + \dot{y}^2 + 2l\dot{y}\dot{\theta}\sin\theta) - mg(y(t) + l\cos\theta). \tag{2.1}$$

$$y(t) = -\frac{l}{\gamma}F_0\sin(\gamma t). \tag{2.2}$$

Solving the first-order Lagrange-Euler equation in θ and $\dot{\theta}$ gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0. \tag{2.3}$$

The differential equation of motion may be written in the form

$$\ddot{\theta} + (F_0 \cos \gamma t \ - \omega_0^2) \sin \theta = 0,\, \omega_0 = \sqrt{\frac{g}{l}}. \tag{2.4}$$

In the following section, we find a analytical solution to the Kapitza pendulum equation.

3 Analytical solution to Kapitza pendulum

Let us consider the initial value problem

$$\ddot{\theta} + (F_0 \cos \gamma t \ - \omega_0^2) \sin(\theta) = 0, \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0 for 0 \le \theta_0 \le \pi \tag{3.5}$$

We make use of the following approximation:
$\sin(\theta) \approx \frac{3\theta^4}{80} - \frac{31\theta^3}{132} + \frac{\theta^2}{18} + \frac{61\theta}{62}$ for $0 \le \theta \le \pi$.
Let $\theta(t) = d + \varphi(t)$, where d is some constant and the function $\varphi \equiv \varphi(t)$ is to be determined. We have:

$$\begin{aligned}
&\ddot{\theta} + (F_0 \cos \gamma t \ - \omega_0^2) \sin(\theta) \approx \\
&\ddot{\theta} + (F_0 \cos \gamma t \ - \omega_0^2) \left(\tfrac{3\theta^4}{80} - \tfrac{31\theta^3}{132} + \tfrac{\theta^2}{18} + \tfrac{61\theta}{62} \right) = \\
&\left(\tfrac{3d^3}{20} - \tfrac{31d^2}{44} + \tfrac{d}{9} + \tfrac{61}{62} \right) G(t)\varphi(t) + \\
&\left(\tfrac{9d^2}{40} - \tfrac{31d}{44} + \tfrac{1}{18} \right) G(t)\varphi(t)^2 + \\
&\left(\tfrac{3d}{20} - \tfrac{31}{132} \right) G(t)\varphi(t)^3 + \tfrac{3}{80} G(t)\varphi(t)^4 + \\
&d \left(\tfrac{3d^3}{80} - \tfrac{31d^2}{132} + \tfrac{d}{18} + \tfrac{61}{62} \right) G(t) + \varphi''(t),
\end{aligned} \tag{3.6}$$

where
$G(t) = F_0 \cos \gamma t \ - \omega_0^2$.

We choose the constant d so that $\frac{3d^3}{80}-\frac{31d^2}{132}+\frac{d}{18}+\frac{61}{62}=0$.
Let $\varphi(t)=c_0\cos(f(t))+c_1\sin(f(t))$. Then

$$\ddot{\theta}+(F_0\cos\gamma t\ -\omega_0^2)\sin(\theta)\approx$$
$$\frac{1}{245520}\left(\begin{array}{c} c_0F_0\left(\begin{array}{c}27621c_0^2d+27621c_1^2d-43245c_0^2-\\ 43245c_1^2+36828d^3-172980d^2+\\ 27280d+241560\end{array}\right)\cos(\gamma t)-\\ 27621c_0^3d\omega_0^2-27621c_0c_1^2d\omega_0^2-\\ 27280c_0d\omega_0^2+245520c_1f\prime\prime(t)-245520c_0f\prime(t)^2+\\ 43245c_0^3\omega_0^2+43245c_0c_1^2\omega_0^2-241560c_0\omega_0^2-\\ 36828c_0d^3\omega_0^2+172980c_0d^2\omega_0^2\end{array}\right)\cos(f(t))+$$
$$\frac{1}{245520}\left(\begin{array}{c} c_1F_0\left(\begin{array}{c}27621c_0^2d+27621c_1^2d-43245c_0^2-\\ 43245c_1^2+36828d^3-\\ 172980d^2+27280d+241560\end{array}\right)\cos(\gamma t)-\\ 27621c_1^3d\omega_0^2-27621c_0^2c_1d\omega_0^2-\\ 27280c_1d\omega_0^2-245520c_0f\prime\prime(t)-\\ 245520c_1f\prime(t)^2+43245c_1^3\omega_0^2+43245c_0^2c_1\omega_0^2-\\ 241560c_1\omega_0^2-36828c_1d^3\omega_0^2+172980c_1d^2\omega_0^2+\end{array}\right)\sin(f(t))+\ \text{h.o.t} \tag{3.7}$$

Equating the coefficients of $\cos(f(t))$ and $\sin(f(t))$ to zero and eliminating the second derivative $f''(t)$ gives the following ordinary differential equation to determine $f=f(t)$:

$$f'(t)^2=\frac{1}{245520}\left(\begin{array}{c}-27621c_0^2d-27621c_1^2d+43245c_0^2+43245c_1^2-\\ 36828d^3+172980d^2-27280d-241560\end{array}\right)\left(\omega_0^2-F_0\cos(\gamma t)\right). \tag{3.8}$$

Solving this ode for $f(0)=0$ gives

$$f(t)=\frac{\sqrt{\begin{array}{c}-27621c_0^2d-27621c_1^2d+43245c_0^2+\\ 43245c_1^2-36828d^3+172980d^2-27280d-241560\end{array}}\sqrt{\omega_0^2-F_0}E\left(\frac{t\gamma}{2}|\frac{2F_0}{F_0-\omega_0^2}\right)}{6\sqrt{1705}\gamma} \tag{3.9}$$

We introduce a correction parameter λ by multiplying the function $f(t)$ by λ so that $\theta(t)=d+c_0\cos(\lambda f(t))+c_1\sin(\lambda f(t))$. The constants c_0 and c_1 are determined from the initial conditions $\theta(0) = \theta_0$ and $\theta'(0) = \dot{\theta}_0$ as follows :

$$c_0 = \theta_0 - d.$$

and

$$-\dot{\theta}_0^2 + \frac{\lambda^2\left(F_0-\omega_0^2\right)}{245520}\begin{pmatrix} 241560 + 27280d - 216225d^2 + \\ 64449d^3 + 86490d\theta_0 - \\ 55242d^2\theta_0 - 43245\theta_0^2 + 27621d\theta_0^2 \end{pmatrix} c_1^2 + \frac{1}{880}(99d - 155)\lambda^2\left(F_0 - \omega_0^2\right)c_1^4 = 0. \tag{3.10}$$

The value of d is defined as $d = 3.152218911433563$.

4 Analysis and Discussion

As a numerical example, we use the following values of the related coefficients

$$\gamma = l = 1, F_0 = 0.2, \theta_0 = 75^\circ,\ \dot{\theta}_0 = 0,\ g = 9.8,\ 0 \le t \le 40. \tag{4.11}$$

Accordingly, the approximate solution reads

$$\theta(t) = 3.15222 - 1.84322\cos(4.94197E(0.5t| - 0.0206186)) + \tag{4.12}$$

$$0.0809394\sin(4.94197E(0.5t| - 0.0206186)) \tag{4.13}$$

Figure 4 illustrates the comparison between the obtained approximate solution (4.12) and the numerical solution using the RK4 method. The optimal λ parameter value is $\lambda = 1.321$. Moreover, the residual error is estimated in the interval $t \in [0, 40]$ and it is given by $L_\infty = 0.1$

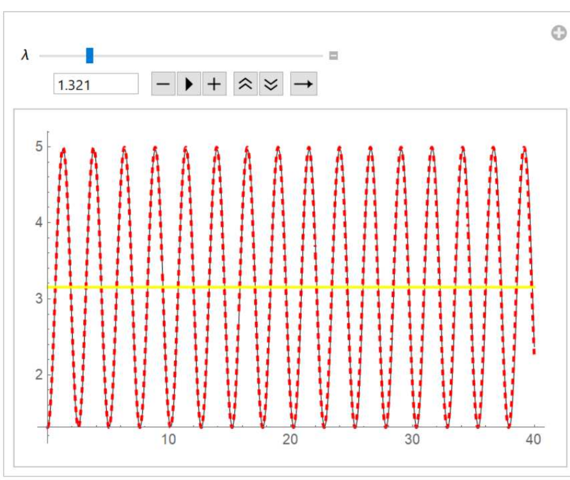

Figure 4: A comparison between the approximate and the RK4 numerical solution. The yellow line corresponds to the equilibrum point $\theta = d$.

5 Conclusions

We solved the differential equation that describes the Kapitza pendulum analytically. In addition, we solved this equation numerically via the fourth-order Runge-Kutta method. We compare the analytical and numerical solutions and observe that the error between the two solutions is very small. Consequently, the obtained analytical solution allows a qualitative analysis to the problem. The proposed methodology should be helpful in solving many nonlinear mathematical and physical problems that have numerous applications in engineering.

Finally, we refer the interested reader to [2]-[5] for other work related to the Kapitza pendulum.

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