International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1665–1675

# Some Properties for the Product and Free Sum of Two Polytopes

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(Received June 1, 2022, Accepted July 13, 2022)

**Abstract**

The product of two polytopes $P$ and $Q$ of orders $m$ and $n$, respectively, yields a polytope of order $m+n$. The resulting polytope is very important in various fields of sciences. The goal of our research is to create a new relationship for the Ehrhart polynomial of the free sum for two polytopes, as well as the product for two cyclic polytopes. We also discuss the roots of all obtained polynomials. Finally, we give applications for these results and provide an algorithm in Matlab.

**Keywords and phrases:** Cyclic polytopes, Ehrhart polynomial, product, free sum, roots of the Ehrhart polynomial.
**AMS (MOS) Subject Classifications**: 51M20, 52B05, 05C30, 51M20.
**ISSN** 1814-0432, 2022, http://ijmcs.future-in-tech.net

# 1 Introduction

Convex polytopes have been studied as essential geometric objects since antiquity. The beauty of their theory is now complemented by its importance in different mathematical fields including algebraic topology, integration theory, algebraic geometry, and linear and combinatorial optimization [1-3]. When a polytope $P$ represents the convex hull of finitely many points in $R^d$, we call it the $V$-representation. If the entire line segment between any two points in $S$ is included in $S$, then the set $S \subseteq R^d$ is said to be convex. The smallest convex set containing a point set $X$ in $R^d$ is said to be the convex hull of $X$. Another form which is called $H$-representation is given as a set $P = \left\{X \in R^d : AX \leq b\right\}$is said to be a polyhedron. Every bounded polyhedron is called a polytope [4, 5]. The polytope $P \subseteq R^d$ is said to be an integral polytope if every vertex of $P$ belongs to $Z$ [6], and the set $tP = \{$tx: x $\in P$, where $t \in$ Z$^+$ } is called the dilated polytope [7], a map $L\left(P, .\right)$ represented by $L\left(P, t\right) = \left|tP \cap Z^d\ \right|$ and $L(P, t)$ can be represented as $L\left(P, t\right) = \sum_{i=0}^{d} c_i t^i$, that is called the Ehrhart polynomial of an integral $d$-polytope of degree $d$. The constant term is one which equals the Euler characteristic of $P$, and $c_d$ is the volume of $P$. The additional coefficients of $L(P, t)$ are difficult to find. Until recently, the method for determining these coefficients was unknown [8].

We derive a general formula that finds the Ehrhart polynomial in high dimension as a free sum for two polytopes together with a discussion of their roots.

# 2 Preliminaries

In this section, we gives some definitions and theorems concerning polytopes.

**Theorem 2.1 [9]**

Let $P$ be a convex integral polygon and let $t$ be a positive integer. The following equality holds $L(P, t) \ = \ c_2 t^2 \ + \frac{1}{2} c_1 t \ \ + 1$,

where $c_2$ is the area of the polytope and $c_1$ is the number of integral points on the boundary of $P$ [10].

**Definition 2.1 [11]** A $d$-polytope with exactly $d + 1$ vertices is called a $d$-dimensional simplex. Equivalently, it is the convex hull of a set of affine independent points in $\mathcal{R}^d$.

**Definition 2.2 [12]** The moment curve in $\mathcal{R}^d$ is defined by $\mathcal{M}$: $\mathcal{R} \rightarrow \mathcal{R}^d$, where $\mathcal{M}\left(\nu\right) = \left(\ \nu, \nu^2, \ldots, \upsilon^d\right)^T \in \mathcal{R}^d$.

**Definition 2.3** [**12**] The convex hull $C_d(v_1, v_2, ..v_n)$ $=conv\{\ \mathcal{M}\left(v_1\right),\ \mathcal{M}\left(v_2\right),\ldots,\ \mathcal{M}\left(v_n\right)\}$ of distinct points $\mathcal{M}\left(v_i\right)$ is the cyclic $C_d(n)$ polytope of dimension $d$ with $n$ vertices, where $n > d$ and $v_1 < v_2 < \ldots v_n$ on the moment curve.

**Theorem 2.2** [**13**] Let $V = \{v_1, v_2, ..v_n\}$ *and* $C_d(V)$ be integral cyclic polytope, where

$L\left(C_d\left(V\right), t\right) = \sum_{j=0}^{d} vol\left(tC_j\left(V\right)\right) = \sum_{j=0}^{d} vol_j\left(C_j\left(V\right)t^j\right)\ldots\ldots\ldots$ (1)

where $vol_j(tC_j\left(V\right))$ is the volume of $tC_j\left(V\right)$ in $j$-dimensional space and $vol_0\left(tC_0\left(V\right)\right) = 1$

**Theorem 2.3** [**13** ]For any integral set $V$ with $n = V = d+1$, (which means that $C_d(n)$ is simplex), we have

$vol\left(tC_d\left(V\right)\right)=\frac{t^d}{d!}\prod_{1\le i<k\le d+1}\left(v_k - v_i\right)\ldots\ldots\ldots\ldots$(2)

**Theorem 2.4** [**14,15**] Let $C_d(V)$ be integral cyclic polytope, where $V = \{v_1, v_2, ..v_n\}$ for $v_i$=$1, 2, \ldots, n$ and $i = 1, 2, \ldots, n$, the following are satisfied:

1. If $C_d(n)$ is not simplex, then it can be decomposed into simplices and the volume can be calculated using equation (2.2).

2. The volume of cyclic the polytope $tC_1\left(V\right)$ in a one-dimensional space in the interval $[v_1$,$v_n]$ is equal to $v_n - v_1$.

**Definition 2.5** [**9**] For two polytopes $P \subseteq R^{dP}$ and $Q \subseteq R^{dQ}$ of dimensions $d_P$ and $d_Q$, the product of $P$ and $Q$ is $P \times Q = \{\left(p, q\right),$ where $p \in P, q \in Q\} \subseteq R^{dP+dP}$.

**Definition 2.6** [**9**] For two polytopes $P \subseteq R^{dP}$ and $Q \subseteq R^{dQ}$ of dimension $d_P$and $d_Q$, the free sum of $P$ and $Q$ is $P \oplus Q = conv\{(0_P \times Q) \cup (P \times 0_Q)\} \subseteq R^{dP+dQ}$.

**Theorem 2.7** [**9**] If $P$ is a $d_P$-dimensional integral polytope in $R^d$ and $Q$ is $d_Q$-dimensional integral polytope in $R^{dQ}$ the Ehrhart polynomial of $P \times Q$ $=L_P\left(t\right)L_Q\left(t\right)$.

**Theorem 2.8** [**16**]

(a) Ehrhart polynomials of lattice $d$-polytopes have roots that are bounded in norm by $1 + (d + 1)!$.

(b) The half-open interval $[-\mathrm{d},\ \lfloor \mathrm{d}/2\rfloor)$ contains every real root of Ehrhart polynomials of $d$-dimensional lattice polytopes.

# 3 Ehrhart Polynomial for the Product of Two Cyclic Polytope

In this section, we introduce the product of the Ehrhart polynomial for two cyclic polytopes which gives the general Ehrhart polynomial for high dimension.
Let $C_2(n)$ be a cyclic polytope. Take $\times$ as a basic definition of a product of two cyclic polytopes.
Let $L(C_2(n), t) = a_2t^2 + a_1t + a_0$ be the Ehrhart polynomial of a cyclic polytope with $n$ vertices. Then
$L_{C_2\times C_2}(t) = L(C_2(n), t).tL(C_2(n), t) = (a_2t^2+a_1t+a_0).(a_2t^2+a_1t+a_0) = b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_0$,
where $b_4 = a_2^2, b_3 = 2(a_2a_1), b_2 = 2(a_2 + a_1^2), b_1 = 2(n-1), b_0 = 1$.
This is given in Table 1. For each $n$ vertices, we obtain the Ehrhart polynomial for the product of two cyclic polytopes.

**Table 1: The Coefficients of the Ehrhart Polynomial**.

| **n** | $\boldsymbol{b_4}$ | $\boldsymbol{b_3}$ | $\boldsymbol{b_2}$ | $\boldsymbol{b_1}$ |
|---|---|---|---|---|
| 3 | 1 | 4 | 6 | 4 |
| 4 | 16 | 24 | 17 | 6 |
| 5 | 100 | 80 | 36 | 8 |
| 6 | 400 | 200 | 65 | 10 |
| 7 | 1225 | 420 | 106 | 12 |
| 8 | 3136 | 784 | 161 | 14 |
| 9 | 7056 | 1344 | 232 | 16 |
| 10 | 14400 | 2160 | 321 | 18 |
| . | . | . | . | . |
| . | . | . | . | . |
| . | . | . | . | . |

### 3.1 Properties of the Roots Ehrhart Polynomial for the Product of Two Cyclic Polytopes

In this section, we give the roots of the Ehrhart polynomials of the product of two cyclic polytopes that satisfy theorem (2.8). The results are obtained using Matlab programming. This is given in Table 2. For each $n$ vertex, the roots of the Ehrhart polynomial for the product of two cyclic polytopes are obtained.

**Table 2: The roots of the Ehrhart polynomial**.

| n | Real parts | Imaginary part |
|---|---|---|
| 3 | -1,-1,-1,-1 | 0 |
| 4 | -3/8, -3/8, -3/8, -3/8 | 253/765, 253/765 |
| 5 | -1/5,-1/5,-1/5,-1/5 | 1176/4801, 1176/4801 |
| 6 | -1/8, -1/8, -1/8, -1/8 | 625/3371, 625/3371 |
| 7 | -3/35, -3/35, -3/35, -3/35 | 743/5100, 743/5100 |
| 8 | -1/16, -1/16, -1/16, -1/16 | 253/2142, 253/2142 |
| 9 | -1/21, -1/21, -1/21, -1/21 | 311/3168, 311/3168 |
| 10 | -3/80, -3/80, -3/80, -3/80 | 133/1598, 133/1598 |

# 4 Ehrhart Polynomial for the Free Sum of Two Polytopes

In this section, we discuss the free sum for two polytopes together with the general form for it. In addition, we give some properties for the Ehrhart polynomial.

**Theorem 4.1** Let $P \subseteq \ R^d$ be a convex lattice polytope such that $P = conv\left((a), (a+s)\right) \subseteq R$, where $a, s \in R$ and $s = 2a, \quad a \neq 0$. Then the Ehrhart polynomial of $P \oplus P$ is equal to $L_P^2(t)$.

**Proof**:

$P = conv\left((a), (a+s)\right) \subseteq R$ is a line segment of length $s$, where $a, s \in R$ and $s = 2a, \quad a \neq 0$, the Ehrhart polynomial of $P$ is equal $st + 1$.

The free sum of $P \oplus P = conv\{(0_P \times P) \cup (P \times 0_P)\} \subseteq R^{dP+dP}$

$= conv\{(0, a), (0, a+s) \cup (a, 0), (a+s, 0)$

The Ehrhart polynomial of $P \oplus P$ is $\mathrm{L}(P \oplus P, \mathrm{t}) = \mathrm{c}_2 \mathrm{t}^2 + \frac{1}{2}\mathrm{c}_1 t + 1$,

where $c_2$ is the area of the polytope and $\mathrm{c}_1$ is the number of integral points on the boundary of P. With $\triangle_2, \ \triangle_1$ being right-angle triangles,

$\mathrm{c}_2 = \triangle_2 - \triangle_1 = \frac{1}{2}(a+s)(a+s) - \frac{a}{2}^2 = \frac{s^2}{2} + as = \frac{(2a)^2}{2} + 2a^2 = \frac{4a^2+4a^2}{2} = 4a^2 = s^2, c_1 = 4s$

$L(P \oplus P, \mathrm{t}) = s^2 t^2 + \frac{1}{2} 4st + 1$.

The Ehrhart polynomial of the free sum of $P \oplus P$ is equal to the Ehrhart polynomial of:

$L_{p \oplus p}(t) = L_p(t) \times L_p(t) = (\ st + 1) \times (\ st + 1) = s^2t^2 + 2st + 1$.

**Example (4.1)**

Let $P = conv\left((1), (3)\right) \subseteq R$ be a line segment of length $s$, where $a \in R, \ \ s = 2, \ \ a \neq 0$. The Ehrhart polynomial of $P$ is $2t + 1$.

The free sum of $P \oplus P = conv\{(0_P \times P) \cup (P \times 0_P)\} \subseteq R^{dP+dP}$
$= conv\{(0,1),(0,3) \cup (1,0),(3,0)\}$
The Ehrhart polynomial of $P \oplus P$ is L($P \oplus P$,t) = $c_2 t^2 + \frac{1}{2} c_1 t + 1$,
where $c_2$ is the area of the polytope and $c_1$ is the number of integral points on the boundary of $P$.
$c_2 = \triangle_2 - \triangle_1 = \frac{s^2}{2} + as = 4, c_1 = 4s = 8 L_{P \oplus P}(t) = 4t^2 + 4t + 1.$
The Ehrhart polynomial of the free sum of $P \oplus P$ is equal to the Ehrhart polynomial of:
$L_{P \oplus P}(t) = L_P(t) \times L_P(t) = (2t+1) \times (2t+1) = 4t^2 + 4t + 1.$
This is different from the general form given in [14].

## 4.1 Properties of the Roots Ehrhart Polynomial for the Free Sum of Two Polytopes

In this section, we give the roots of the Ehrhart polynomials of the free sum of two polytopes $P = conv((a),(a+s)) \subseteq R$ satisfying theorem (2.8). The results are obtained using Matlab programming. This is given in Table 3. The general formula of roots of the Ehrhart polynomials of the free sum of two polytopes $P = conv((a),(a+s)) \subseteq R$ is $1/s$.

**Table 3: The roots of the Ehrhart polynomial**

| a | s | d_4 | d_3 | d_2 | d_1 |
|---|---|---|---|---|---|
| 1 | 3 | 16 | 32 | 24 | 8 |
| 2 | 6 | 256 | 256 | 96 | 16 |
| 3 | 9 | 1296 | 864 | 216 | 24 |
| 4 | 12 | 4096 | 2048 | 384 | 32 |
| 5 | 15 | 10000 | 4000 | 600 | 40 |

## 4.2 Product of two Polytopes in $n$ Dimensions

One can take the Ehrhart polynomial obtained from subsection 4.1 and, for a high dimension that is $L_P \times L_P$, get the polytopes of dimension four. Similarly, for the product of two polytopes of dimension four, one gets the Ehrhart polynomial of dimension eight, and so on.
Let $P = conv((a),(a+s)) \subseteq R$, where $a, s \in R$ and $s = 2a, \quad a \neq 0$

$$L_P = st + 1.$$

Let $L_{P \oplus P}(t) = L_Q(t) = s^2 t^2 + 2st + 1.$

Then $L_Q(t) \times L_Q(t) = (s^2t^2 + 2st + 1) \times (s^2t^2 + 2st + 1)$

$$= s^4t^4 + 4s^3t^3 + 6s^2t^2 + 4st + 1.$$

This is given in Table 4. For each $n$ vertices, we obtain the Ehrhart polynomial for the product of two polytopes.

**Table 4:** The coefficients of the Ehrhart polynomial.

| a | s | $d_4$ | $d_3$ | $d_2$ | $d_1$ |
|---|---|---|---|---|---|
| 1 | 3 | 16 | 32 | 24 | 8 |
| 2 | 6 | 256 | 256 | 96 | 16 |
| 3 | 9 | 1296 | 864 | 216 | 24 |
| 4 | 12 | 4096 | 2048 | 384 | 32 |
| 5 | 15 | 10000 | 4000 | 600 | 40 |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| . | . | . | . | . | . |

## 4.3 Properties of the Roots of the Ehrhart Polynomial for the product of Two Polytopes

In this section, we give the roots of the Ehrhart polynomial obtained from subsection 4.1 to satisfy theorem (2.8). The results are obtained using Matlab programming. This is given in Table 5.

**Table 5:** The coefficients of the Ehrhart polynomial.

| a | a+s | Real part | Imaginary part |
|---|---|---|---|
| 1 | 3 | -4137/8273,4137/8273, | 8/132345, 5/82729 |
| 2 | 6 | -2068/8271,-2068/8273 | 4/132345, 8264733 |
| 3 | 9 | -1700/10199,-1/6 | 3/183698,0,0 |
| 4 | 12 | -1034/8271,-1034/8273 | 2/132345, 2/132345 |
| 5 | 15 | -649/6489, -649/6489 | 14/908951, 4259773 |

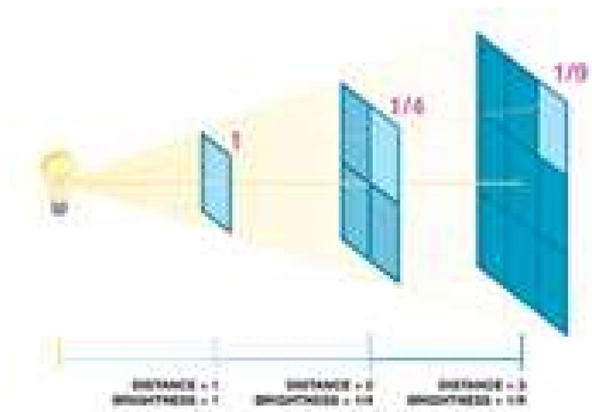


Figure 1: Inverse-square law

# 5 Application of the Product and Free Sum of Two Polytopes

Below are some examples of applications that use strong points along with their figures.

1. **A brightly lit area** [18]

The position and direction of light in space determine its distribution. Radiance is a useful unit for measuring light division in space; it is described as the power (amount of energy per unit time) flowing at a certain position in a specific direction, per unit area perpendicular to the direction of travel, per unit solid angle, at a specific point. The amount of light radiated from a single location is known as radiance (into a unit solid angle, from a unit area)

Radiance = Power / (foreshortened area of solid angle)

2. **Radiant intensity**

Radiance is defined as the amount of energy going at some point in a given direction per unit area perpendicular to that direction, per unit solid angle [19].

**3. 3D-Dimension Cinema**

In 3D theaters, the capture of an image is done uniquely. During the filming of the motion pictures, two cameras are used. Both cameras capture the same demonstration but at a somewhat unusual point. Our two eyes are represented by the angle and cameras. The camera's solid angle distinction is comparable to that of a human eye. The two cameras independently

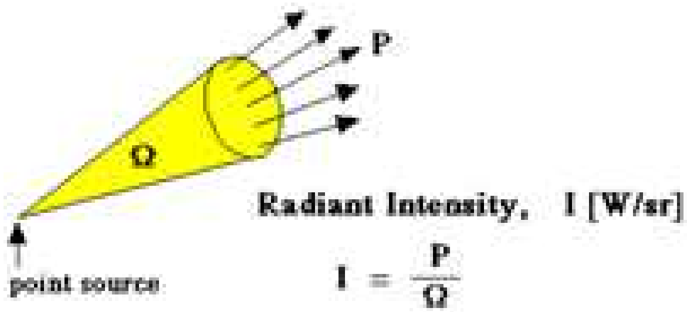


Figure 2: Radiation

Figure 3: 3D Cinema

capture two images of the same demonstration, which are similarly framed by differing polarization of light.

Two various wellsprings of light are on par with two diversely captivated lights. These two photographs are carefully tucked away. These images are duplicated and anticipated on the screen to the point where two sources are expected with two contrastingly captivated lights in a cinema theater. The expected surface is designed to avoid changing the polarization on reflection. As a result, the reflected light has two spellbound light sources.

On the off chance that we use Polaroid glasses, this is comparable to discrete sources. The Polaroid glasses block one type of polarization while allowing the other. This framework satisfies our basic need for two sources of light from the same object to appear three-dimensional to our different eyes.

# 6 Conclusion

In this paper, we proved that the free sum of two convex lattice polytopes is equal to the product of the Ehrhart polynomial of those polytopes. All real roots of Ehrhart polynomials of two product cyclic polytopes as well as all produced polynomials reside in the half-open interval $[d, d/2]$. Matlab programming was used to acquire the findings.

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