International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1513–1528

On some properties of power Garima-Topp Leone power Lomax distribution with application to lifetime data

Issaraporn Thiamsorn[1], **Satsayamon Suksaengrakcharoen**[2]

[1]Department of Mathematics and Computer Science
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi
Pathum Thani, 12110, Thailand

[2]School of Science
University of Phayao
Phayao, 56000, Thailand

email: t_issaraporn@rmutt.ac.th, satsayamon.su@up.ac.th

(Received April 3, 2022, Accepted May 24, 2022)

Abstract

In this article, we propose a new lifetime distribution; namely, the power Garima-Topp Leone power Lomax (PG-TLpL) distribution which is derived from the power Garima-G family of distributions when using the type II Topp Leone power Lomax (TLpL) distribution as a baseline distribution. The proposed distribution has three submodels: the power Garima- Topp Leone Lomax, Garima-Topp Leone power Lomax, and the Garima-Topp Leone Lomax distribution. In practice, the proposed distribution has various shapes; i.e., symmetric, right-skewed, left-skewed, and reversed-J shaped. We introduce properties of the proposed distribution. We use the maximum likelihood estimation (MLE) to estimate the parameters of the proposed distribution. In addition, we examine the efficiency and importance of the PG-TLpL distribution through real data sets. Our results show that the PG-TLpL distribution is a flexible alternative to describe the lifetime data.

Key words and phrases: Power Garima generated family, power Lomax distribution, power Garima-Topp Leone power Lomax, MLE, lifetime data.
AMS (MOS) Subject Classifications: 62-04.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

1 Introduction

Statistical modeling of continuous proportions have received special attention in the last few years. Flexible distributions are obtained from extending the classical distributions by introducing additional shape parameters to the baseline distribution. By expanding common families of continuous distributions, several classes of distribution have been created. Many researchers proposed the generalized distributions, including the beta-generated (B-G) family [1], gamma-G (type I) family [2], Kumaraswamy-G family [3], generalized beta-G family [4], gamma-G family, beta exponential-G family, Weibull-G family [5], exponentiated-G [6], Weibull-G [7], the Logistic-G [8], odd-generalized exponential-G [9], generalized Weibull-G [10], additive Weibull-G [11], Kumaraswamy Weibull-G [12], generalized transmuted-G [13], Gompertz-G [14], beta Weibull-G [15], new Weibull-G [16], and transmuted Gompertz-G [17].

In 2021, Aryuyuen et al. [18] suggested the power Garima-generated (PG-G) family of distributions whose cumulative density function (cdf) is

$$F_{\mathrm{PG-G}}(x) = 1 - \left[1 + \frac{\beta}{2+\beta}\left(\frac{G^{\alpha}(x;\xi)}{1-G^{\alpha}(x;\xi)}\right)^{\lambda}\right] \times \exp\left\{-\beta\left(\frac{G^{\alpha}(x;\xi)}{1-G^{\alpha}(x;\xi)}\right)^{\lambda}\right\}, x > 0, \tag{1.1}$$

where $\alpha > 0$, $\lambda > 0$, $\beta > 0$, and $G(x;\xi)$ is the cdf of any existing (baseline) distributions with the parameter vector of ξ. Its corresponding probability density function (pdf) is

$$f_{\mathrm{PG-G}}(x) = \frac{\alpha\lambda\beta g(x;\xi)G^{\alpha-1}(x;\xi)}{(2+\beta)[1-G^{\alpha}(x;\xi)]^2}\left[1+\beta+\beta\left(\frac{G^{\alpha}(x;\xi)}{1-G^{\alpha}(x;\xi)}\right)^{\lambda}\right] \times \left(\frac{G^{\alpha}(x;\xi)}{1-G^{\alpha}(x;\xi)}\right)^{\lambda-1}\exp\left\{-\beta\left(\frac{G^{\alpha}(x;\xi)}{1-G^{\alpha}(x;\xi)}\right)^{\lambda}\right\}, \tag{1.2}$$

where $g(x;\xi)$ is the pdf of any baseline distribution with the parameter vector of ξ. For $\lambda = 1$, we have the pdf of any baseline distribution with the parameter vector of α and β. The reasons that the PG-G family of distributions were selected are:

(i) It is a prominent method of introducing an additional parameter(s) to generate an extended version of the baseline distribution;

(ii) It improves the characteristics of the traditional distributions;
(iii) It makes the kurtosis more flexible compared to the baseline distribution;
(iv) It generates distributions that have the pdf with various shapes, such as symmetric, right-skewed, left-skewed, and reversed-J shaped;
(v) It defines special models with all types of hazard rate function;
(vi) It defies special models having a closed-form for cdf, survival function as well as hazard rate function; (vii) It provides the better fit of data and distribution than other generated distributions that have the same or higher number of parameters [18].
Moreover, Aryuyuen et al. developed PG-G distributions (e.g., the PG-Fréchet, PG-Weibull, and PG-Lindley distributions) to demonstrate the efficiency and importance of the newly generated family through examining real data sets. The results show that the PG-G distribution is a better model compared to the baseline distribution to these data sets (e.g., the breaking stress of carbon fibers, the failure times of aircraft windshields, and the time to failure of the turbocharger) [18].

The main objective of this work is to create a new PG-G distribution based on the type II Topp Leone power Lomax distribution to be the baseline distribution. The remainder of this paper is organized as follows: In Section 2, we discuss the TLpL distribution. In Section 3, we propose a new PG-G distribution and its special sub-models. In Section 4, we give some statistical properties of the proposed distribution. In Section 5, we provide an estimation of the model parameters by using the maximum likelihood method and, in Section 6, we illustrate its applications. Finally, we conclude our paper in Section 7.

2 The type II Topp Leone power Lomax distribution

In this section, the type II Topp Leone power Lomax (TLpL) distribution, proposed by Aryuyuen and Bodhisuwan [19] in 2020, is presented. Its cdf and pdf are as follows:

$$G(x;a,b,c,d) = 1-\left\{d^b(d+x^c)^{-b}\left[2-d^b(d+x^c)^{-b}\right]\right\}^a, \tag{2.3}$$

$$\begin{aligned} g(x;\gamma,\theta,\upsilon) &= 2abcd^b x^{c-1}(d+x^c)^{-b-1}\left[1-d^b(d+x^c)^{-b}\right] \\ &\quad \times\left\{d^b(d+x^c)^{-b}\left[2-d^b(d+x^c)^{-b}\right]\right\}^{a-1}, \end{aligned} \tag{2.4}$$

where $0 < x < \infty$, $a, b, c, d > 0$. For $c = 1$, the TLpL distribution reduces to the type II Topp Leone Lomax (TLL) distribution. The pdf plots of the TLpL distribution are shown in Figure 1. The shape of the TLpL pdf is a unimodal shape for $c \geq 1$ and a decreasing shape for $c < 1$ [19].

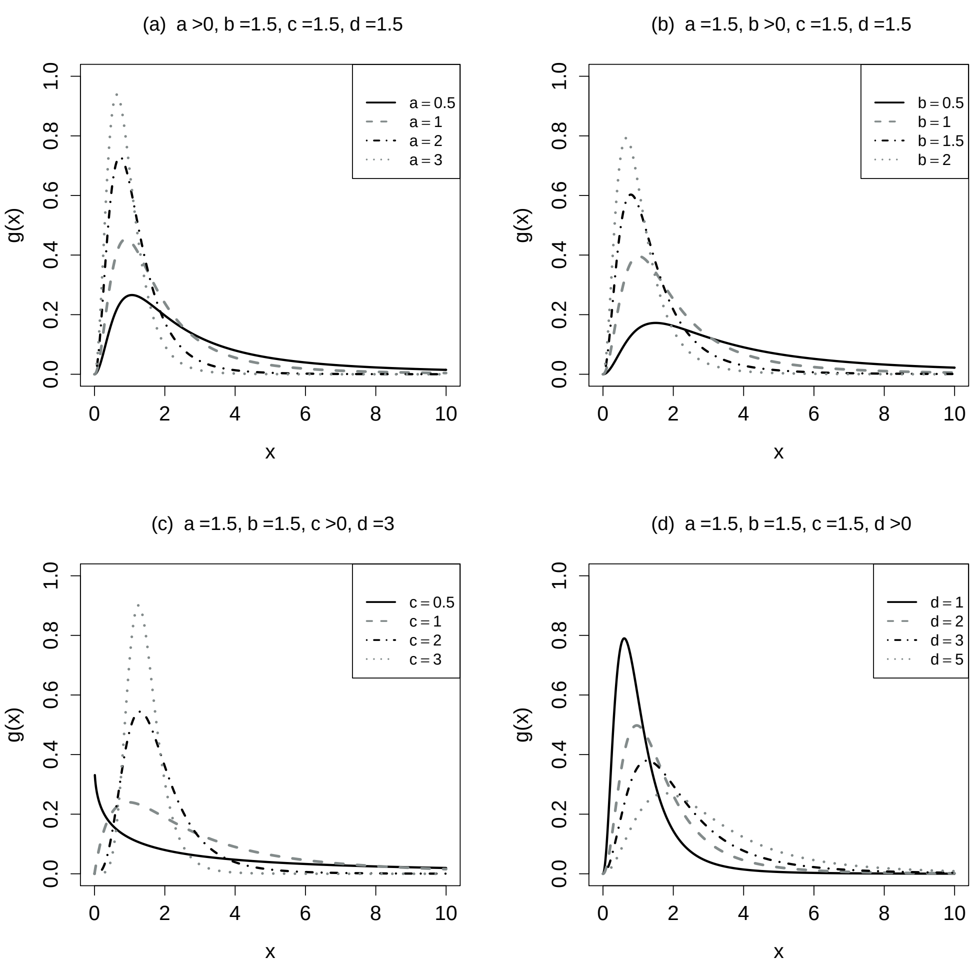

Figure 1: The pdf plots of the TLpL distribution with some specified values of a, b, c, and d.

3 Power Garima-Topp Leone power Lomax distribution

In this section, we define a new PG-G distribution, the so-called a power Garima-Topp Leone power Lomax (PG-TLpL) distribution. The PG-TLpL distribution is a PG-G family of distributions by using the TLpL distribution as the baseline distribution. By replacing the cdf in (2.3) as in the equation (1.1), the cdf of the PG-TLpL distribution is given by

$$F(x;\mathbf{\Theta}) = \mathbf{1} - \left(\mathbf{1} + \frac{\beta}{\mathbf{2}+\beta}\delta^{\lambda}_{(\alpha,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}(\mathbf{x})\right)\exp\left\{-\beta\delta^{\lambda}_{(\alpha,\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})}(\mathbf{x})\right\}. \quad (3.5)$$

Its corresponding pdf is

$$\begin{aligned} f(x;\mathbf{\Theta}) = {} & \frac{2abcd^b\alpha\lambda\beta}{(2+\beta)}\left\{\frac{x^{c-1}(d+x^c)^{-b-1}\left[1-d^b(d+x^c)^{-b}\right]\delta_0^{a-1}}{(1-\delta_0^a)\left[1-(1-\delta_0^a)^{\alpha}\right]}\right\} \\ & \times\left[1+\beta+\beta\delta^{\lambda}_{(\alpha,a,b,c,d)}(x)\right]\delta^{\lambda}_{(\alpha,a,b,c,d)}(x) \\ & \times\exp\left\{-\beta\delta^{\lambda}_{(\alpha,a,b,c,d)}(x)\right\}, \end{aligned} \quad (3.6)$$

where $x > 0$, $\mathbf{\Theta} = (\alpha, \beta, \lambda, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, and

$$\delta_{(\alpha,a,b,c,d)}(x) = \frac{(1-\delta_0^a)^{\alpha}}{1-(1-\delta_0^a)^{\alpha}}, \delta_0 = d^b(d+x^c)^{-b}\left[2-d^b(d+x^c)^{-b}\right].$$

Even though this distribution has seven parameters, that is not a short-come as, nowadays, technology plays its role. Therefore, we can use the software to easily estimate the parameters. Figure 2 shows the pdf plots of the PG-TLpL distribution with the specified value of parameters. This comes in a variety of forms, including, symmetric, right-skewed, left-skewed, and reversed-J shaped. The following are the special sub-models in the proposed distribution:

(i) If $c = 1$, then the PG-TLpL distribution reduces to the power Garima-Topp Leone Lomax (PG-TLL) distribution with the positive parameters $\alpha, \beta, \lambda, a, b$, and d.

(ii) If $\lambda = 1$, then the PG-TLpL distribution reduces to the power Garima-Topp Leone Lomax (PG-TLL) distribution with the positive parameters α, β, a, b, c, and d.

(iii) For $\lambda = 1$ and $c = 1$, then the PG-TLpL distribution reduces to the Garima-Topp Leone Lomax distribution with the positive parameters α, β, a, b, and d.

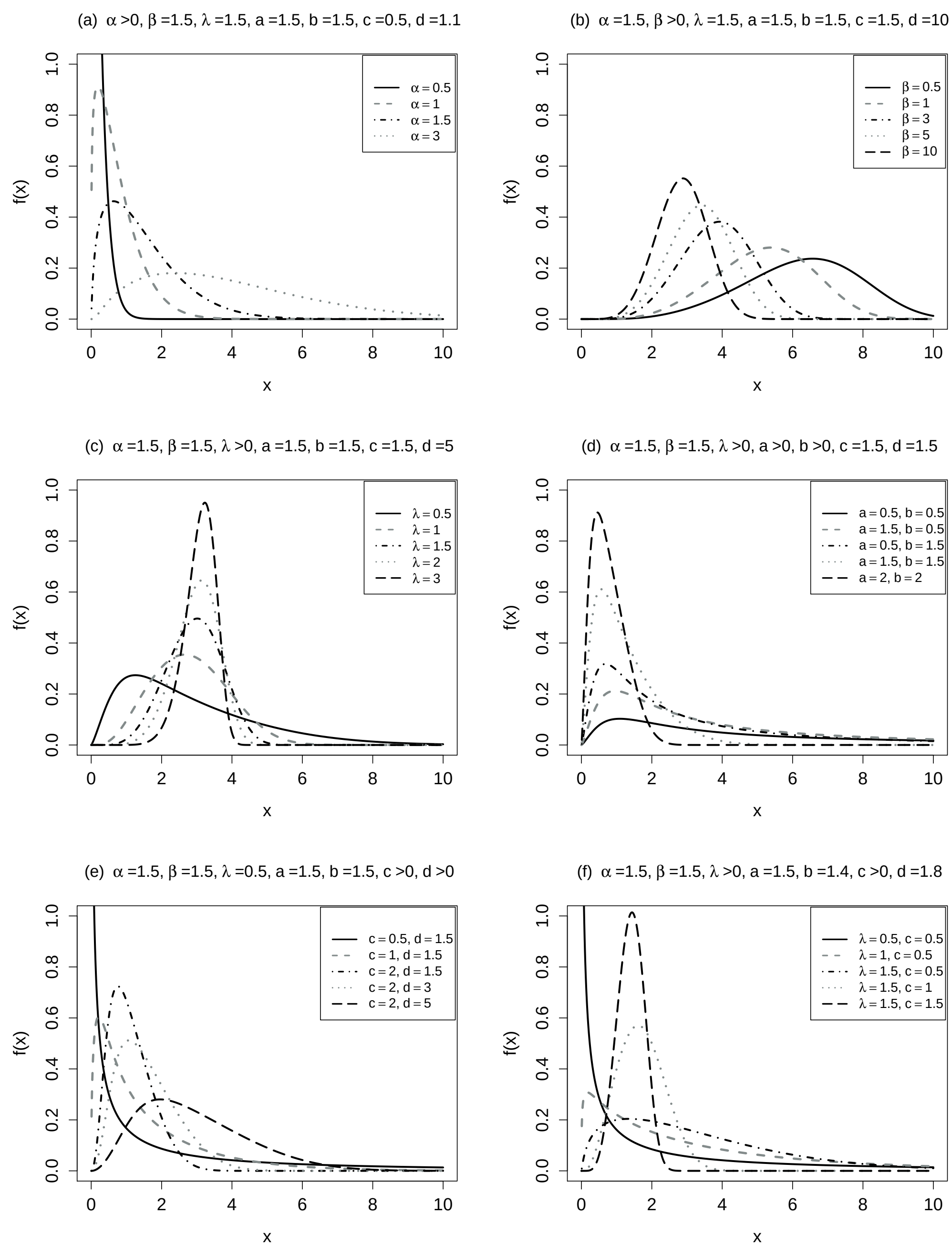

Figure 2: The pdf plots of the PG-TLpL distribution with some specified values of $\alpha, \beta, \lambda, a, b, c$, and d.

4 Statistical properties

In this section, we derive the quantile function, survival and hazard rate functions, and an algorithm of the PG-TLpL generating a random number.

4.1 The quantile function

The quantile function of the PG-G family of distributions is obtained by inverting equation (1.1) as follows (see [18]):

$$Q_{\mathrm{PG-G}}(u) = G^{-1}\left\{\left\{1+\left\{-\beta^{-1}\left\{W_{-1}\left\{-(1-u)(2+\beta)\times\exp(-2-\beta)\right\}+\beta+2\right\}\right\}^{\frac{-1}{\lambda}}\right\}^{\frac{-1}{\alpha}};\xi\right\} \tag{4.7}$$

with $0 < u < 1$, where W_{-1} denotes the negative branch of the Lambert W function (see [20], [21]); i.e., $W(z)\exp[W(z)] = z$, where z is a complex number and $G^{-1}(\cdot)$ is the quantile function of the baseline distribution. From the cdf of the TLpL distribution as (2.3), its inverse function is (see [19])

$$G^{-1}(u) = d^{1/c}\left\{\left[1-\left(1-(1-u)^{1/a}\right)^{1/2}\right]^{-1/b}-1\right\}^{1/c}, 0 < u < 1. \tag{4.8}$$

The quantile function of the PG-TLpL distribution is

$$Q_{\mathrm{PG-TLpL}}(u) = d^{1/c}\left\{\left[1-\left(1-\left[1-\left(1+\left[-\frac{1}{\beta}\left(W_{-1}\left\{-(1-u)(2+\beta)\right\}\right.\right.\right.\right.\right.\right.\right. \\ \left.\left.\left.\left.\left.\left.\left.\times\exp(-2-\beta)+\beta+2\right)\right]^{-1/\lambda}\right)^{-1/\alpha}\right]^{1/a}\right)^{1/2}\right]^{-1/b}-1\right\}, \tag{4.9}$$

where U is a uniform random variable on interval $(0, 1)$.

4.2 The survival and hazard rate functions

Let X be a continuous random variable with cdf in (3.5). Its survival function or reliability function is

$$S(x;\Theta) = \left(1+\frac{\beta}{2+\beta}\delta^{\lambda}_{(\alpha,a,b,c,d)}(x)\right)\exp\left\{-\beta\delta^{\lambda}_{(\alpha,a,b,c,d)}(x)\right\}, x > 0. \tag{4.10}$$

Its associate hazard function is

$$\begin{aligned} h(x;\Theta) &= \frac{2abcd^b\alpha\lambda\beta}{(2+\beta)}\left\{\frac{x^{c-1}(d+x^c)^{-b-1}\left[1-d^b(d+x^c)^{-b}\right]\delta_0^{a-1}}{(1-\delta_0^a)\left[1-(1-\delta_0^a)^\alpha\right]}\right\} \\ &\times\frac{\left[1+\beta+\beta\delta^\lambda_{(\alpha,a,b,c,d)}(x)\right]\delta^\lambda_{(\alpha,a,b,c,d)}(x)}{\left(1+\frac{\beta}{2+\beta}\delta^\lambda_{(\alpha,a,b,c,d)}(x)\right)}, x>0. \end{aligned} \tag{4.11}$$

Figure 3 shows the plots of survival and hazard functions of the PG-TLpL distribution with the specified value of parameters.

4.3 Rényi Entropies

Statistical entropy is a probabilistic measurement of ignorance of the outcome of a random experiment and is a measurement of a reduction in that uncertainty. Entropy of X with pdf $f(x)$ is a measurement of variation of the uncertainty [22]. Rényi entropy is defined by

$$I_R(\zeta)=\frac{1}{1-\zeta}\log\left[\int f^\zeta(x)dx\right] \tag{4.12}$$

where, $\zeta>0$. Rényi entropy of the PG-TLpL distribution is

$$\begin{aligned} I_R(\zeta,\Theta) &= \frac{1}{1-\zeta}\zeta\log\left(\frac{2abcd^b\alpha\lambda\beta}{2+\beta}\right) \\ &\times\int_0^\infty\left\{\zeta\log\left\{\frac{x^{c-1}(d+x^c)^{-b-1}\left[1-d^b(d+x^c)^{-b}\right]\delta_0^{a-1}}{(1-\delta_p^a)\left[1-(1-\delta_0^a)^\alpha\right]}\right.\right. \\ &\times\left[1+\beta+\beta\delta^\lambda_{(\alpha,a,b,c,d)}(x)\right]\delta^\lambda_{(\alpha,a,b,c,d)}(x) \\ &\times\exp\left\{-\beta\delta^\lambda_{(\alpha,a,b,c,d)}(x)\right\}, \end{aligned} \tag{4.13}$$

where a parameter vector $\Theta=(\alpha,\lambda,\beta,a,b,c,d)$.

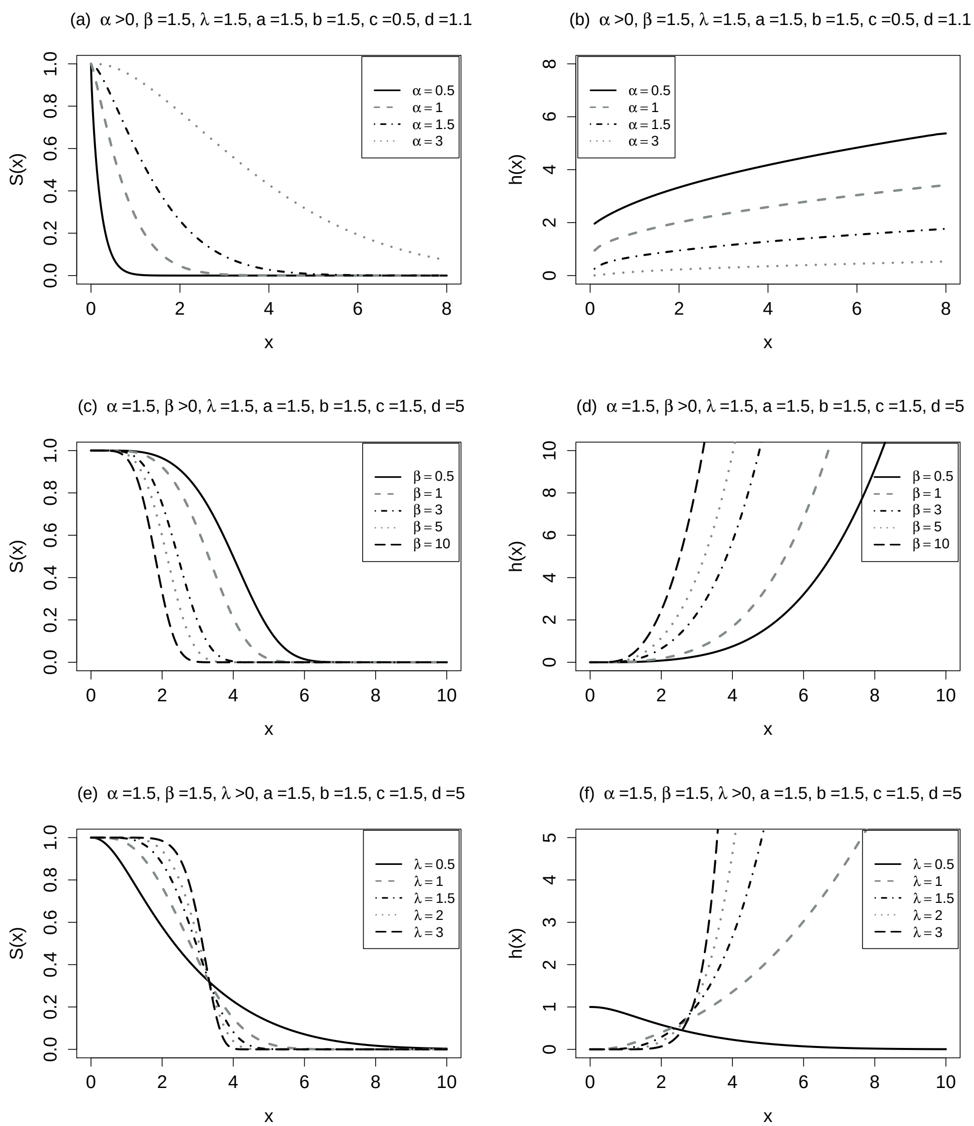

Figure 3: Plots of the survival and hazard functions of the PG-TLpL distribution with some specified values of α,β, λ,a,b,c, and d.

4.4 Random variate generation

We can generate a random variate X from the PG-TLpL distribution using the following algorithm.

Step 1: Generate T with the PG-G distribution by using the quantile function in equation (4.7) as follows:

(i) Generate U from the uniform distribution on interval $(0,1)$.

(ii) Set T as follows:

$$T = \left\{ 1 + \left\{ -\frac{1}{\beta}\{W_{-1}[-(1-U)(2+\beta)\exp(-2-\beta)] + \beta + 2\} \right\}^{-1/\lambda} \right\}^{-1/\alpha}.$$

Step 2: Generate X by using the quantile function in equation (4.9), by setting

$$\begin{aligned} X &= d^{1/c}\left\{ \left[1 - \left(1 - \left[1 - \left(1 + \left[-\frac{1}{\beta}\left(W_{-1}\left\{ -(1-U)(2+\beta) \right\} \right.\right.\right.\right.\right.\right.\right. \\ &\qquad \left.\left.\left.\left.\left.\left.\left. \times \exp(-2-\beta) + \beta + 2 \right) \right]^{-1/\lambda} \right)^{-1/\alpha} \right)^{1/2} \right]^{-1/b} - 1 \right\} \end{aligned} \tag{4.14}$$

For calculating $W_{-1}(z)$ in equation (4.7) use `W(z,branch=-1)` in the **LambertW** contribution package in R [23].

5 Parameter estimation

In this section, we develop the maximum likelihood estimators (MLE) of the model parameters $\Theta = (\alpha, \lambda, \beta, a, b, c, d)$ of the PG-TLpL family. Let $X_1, \ldots, X_n$ be a random sample from the PG-TLpL family with observed values $x_1, \ldots, x_n$. The log-likelihood function of the pdf (3.6) is $\ell(\Theta)$, where

$$\begin{aligned}
\ell(\Theta) &= \log \prod_{i=1}^{n} f(x_i; \Theta) = n \log 2 + n \log(a) + n \log b + n \log c + nb \log d \\
&+ n \log \lambda + n \log \beta - n \log(2+\beta) + \sum_{i=1}^{n} \left\{ -\beta \delta^{\lambda}_{(\alpha,a,b,c,d)}(x_i) \right\} \\
&+ n \log \alpha + \sum_{i=1}^{n} \log \left\{ \frac{x_i^{c-1} (d + x_i^c)^{-b-1} \left[1 - d^b (d + x_i^c)^{-b}\right] \delta_0^{a-1}}{(1-\delta_0^a)\left[1 - (1-\delta_0^a)^{\alpha}\right]} \right\} \\
&+ \sum_{i=1}^{n} \log \left[1 + \beta + \beta \delta^{\lambda}_{(\alpha,a,b,c,d)}(x_i)\right] + \sum_{i=1}^{n} \log \delta^{\lambda}_{(\alpha,a,b,c,d)}(x_i).
\end{aligned}$$

The MLE of $\hat{\alpha}, \hat{\lambda}, \hat{\beta}, \hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ for the parameters Θ are calculated by setting the partial derivative $\ell(\Theta)$ with respect the parameter is equal zero; i.e.,

$$\frac{\partial \ell(Theta)}{\partial \alpha} = 0, \frac{\partial \ell(\Theta)}{\partial \beta} = 0, \frac{\partial \ell(\Theta)}{\partial \lambda} = 0, \frac{\partial \ell(\Theta))}{\partial a} = 0,$$

$$\frac{\partial \ell(\Theta))}{\partial b} = 0, \frac{\partial \ell(\Theta))}{\partial c} = 0, \frac{\partial \ell(\Theta))}{\partial d} = 0.$$

These equations, however, are non-linear. Therefore, we solve them simultaneously using a numerical procedure with the Newton-Raphson method. The *nlm* function in the **stats** package, contribution package in R [23] is used to find the estimated parameters.

6 Applications

In this section, we demonstrate the flexibility and the potentiality of the PG-TLpL distribution through two real data sets. The following are some descriptive statistics for all data sets. The first data set (Data I) is the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital between 1929 and 1938, which Lee [24] discussed : 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0,

Table 1: Parameter estimates and statistic values of each models for Data I.

	Distributions	
Parameter estimates	PG-TLL	PG-TLpL
$\hat{\alpha}$	0.2964	1.1569
$\hat{\beta}$	0.0106	0.0386
$\hat{\lambda}$	1.0975	2.9726
$\hat{a}$	1.2211	1.8250
$\hat{b}$	0.8929	1.5737
$\hat{c}$	-	0.2072
$\hat{d}$	1.1162	1.4746
$-\log L$	578.8625	578.9611
KS (p-value)	0.0494 (0.9292)	0.0487 (0.9367)

44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

The second data set (Data II) is the time to failure (103h) of turbocharger of one type of engine, which is designed by Xu et al. [25]: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The maximum likelihood method for the proposed model was estimated by the model parameters, as described in Section 6. These real data sets were used to determine the estimated parameters of each distribution. The values of MLE of each distribution were obtained by using the nlm function in the stats package, contributed package in R [23]. In each application, the t of the PG-TLpL distribution and the PG-TLL distribution were compared.

A goodness of fit test of the distance between the empirical distribution function $\mathrm{F}_n(x)$ of the samples and the cdf of the reference distribution $\mathrm{F}_0(x)$ are considered by using the Kolmogorov-Smirnov (K-S) test. The model provides the smallest values of K-S statistic. Therefore, it is the best model for fitting data. The results of the MLE, and K-S values for fitting distributions of each data set are shown in Tables 1-2.

For two different data sets, the PG-TLpL gives a smaller K-S than the PG-TLL distribution. These results indicate that the PG-TLpL distribution is appropriate for these data sets. In addition, in Figure 4, we give the probability plot (P-P) of the PG-TLpL distribution. The results suggest that the PG-TLpL distribution is closely related to these data. Therefore,

Table 2: Parameter estimates and statistic values of each models for Data II.

Parameter estimates	Distributions	
	PG-TLL	PG-TLpL
$\hat{\alpha}$	0.1594	0.6699
$\hat{\beta}$	0.0010	0.0020
$\hat{\lambda}$	1.4371	0.4951
$\hat{a}$	2.6841	2.6237
$\hat{b}$	1.5368	2.2320
$\hat{c}$	-	1.3824
$\hat{d}$	3.0345	1.2660
$-\log L$	80.7546	81.4453
KS (p-value)	0.0957 (0.8573)	0.0948 (0.8651)

the proposed distribution is a flexible alternative to describe the lifetime data.

7 Discussion and conclusion

In this paper, we proposed a new lifetime distribution; namely, the power Garima-Topp Leone power Lomax (PG-TLpL) distribution, created from the PG-G family of distributions by utilizing the Topp Leone power Lomax (TLpL) distribution as a baseline distribution. The proposed distribution has three sub-models: the power Garima-Topp Leone Lomax (PG-TLL), Garima-Topp Leone power Lomax, and Garima-Topp Leone Lomax distributions. In practice, the proposed distribution has various shapes; i.e., symmetric, right-skewed, left-skewed, and reversed-J shaped. We gave some properties of the new distribution, including quantile function, survival and hazard rate functions, and Rényi Entropies. Generating a random variable of the PG-TLpL distribution was proposed. We developed the maximum likelihood estimation for estimating the parameters of the PG-TLpL distributions. Moreover, we investigated the efficiency and importance of the PG-TLpL distribution examined through real data sets. Results show that the PG-TLpL distribution was an flexible alternative to describe the lifetime data.

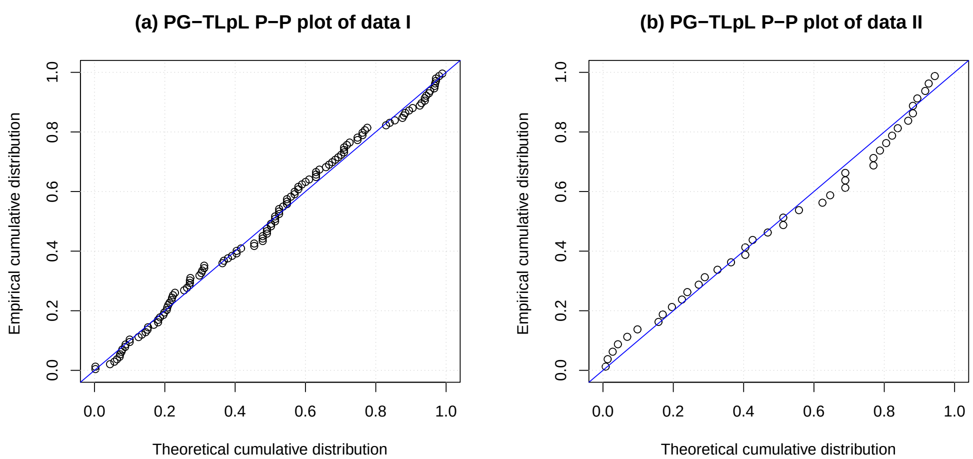

Figure 4: The P-P plots of the PG-TLpL distribution for each real data sets.

Acknowledgment. The author would like to thank the anonymous reviewers for their comments and suggestions.

References

[1] N. Eugene, C. Lee, F. Famoye, Beta-normal distribution and its applications Commun. Stat. - Theory Methods, **31,** no. 4, (2002), 497–512.

[2] K. Zografos, N. Balakrishnan, On families of beta-and generalized gamma-generated distributions and associated inference, Stat. Methods, **6,** no. 4, (2009), 344–362.

[3] G. M. Cordeiro, M. de Castro, A new family of generalized distributions, J. Stat. Comput. Simul, **81,** no. 7, (2011) 883–898.

[4] C. Alexander, G. M. Cordeiro, M. M. Ortega, J. M. Sarabia, Generalized beta-generated distributions, Comput. Stat. Data. Anal., **56,**, (2012), 1880–1897.

[5] A. Alzaatreh, C. Lee, F. Famoye, A new method for generating families of continuous distributions, Metron, **71**, (2013), 63–79.

[6] G. M. Cordeiro, E. M. Ortega, D. C. da Cunha, The exponentiated generalized class of distributions, J. Data. Sci., **11,** no. 1, (2013), 1–27.

[7] M. Bourguignon, R. B. Silva, G. M. Cordeiro, The Weibull-G family of probability distributions, J. Data. Sci., **12**, (2014), 53–68.

[8] H. Torabi, N. H. Montazeri, The logistic-uniform distribution and its applications, Commun. Stat. - Simul. Comput., **43,** no. 10, (2014), 2551–2569.

[9] M. H. Tahir, G. M. Cordeiro, M. Alizadeh, M. Mansoor, M. Zubair, G. G. Hamedani, The odd generalized exponential family of distributions with applications, J. Stat. Distrib. Appl., **2,** no. 1, (2015), 1–28.

[10] G. M. Cordeiro, E. M. Ortega, T. G. Ramires, A new generalized Weibull family of distributions: mathematical properties and applications, J. Stat. Distrib. Appl,, **2,** no. 1, (2015), 1–25.

[11] A. S. Hassan, S. E. Hemeda, A new family of additive Weibull-generated distributions, Int. J. Math. Appl., **4,** no. 2, (2016), 151–164.

[12] A. S. Hassan, M. Elgarhy, Kumaraswamy Weibull-generated family of distributions with applications, Adv. Appl. Stat., **48,** no. 3, (2016), 205–239.

[13] Z. M. Nofal, A. Z. Afify, H. M .Yousof, G. M. Cordeiro, The generalized transmuted-G family of distributions, Commun. Stat. - Theory Methods, **46,**, no. 8, (2017), 4119–4136.

[14] M. Alizadeh, G. M. Cordeiro, L. G. B. Pinho, I. Ghosh, The Gompertz-G family of distributions, J. Stat. Theory Pract., **11**, no. 1, (2017), 179–207.

[15] H. M. Yousof, M. Rasekhi, A. Z. Afify, I. Ghosh, M. Alizadeh, G. G. Hamedani, The beta Weibull-G family of distributions: theory, characterizations and applications, Pakistan J. Stat., **33,** no. 2, (2017), 95–116.

[16] Z. Ahmad, M. Elgarhy, G. G. Hamedani, A new Weibull-X family of distributions: properties, characterizations and applications, J. Stat. Distrib. Appl., **5,** no. 1, (2018), 1–18.

[17] H. Reyad, F. Jamal, S. Othman, G. G. Hamedani, The transmuted Gompertz-G family of distributions: properties and applications, Tbil. Math. J. , **11,** no. 3, (2018), 47–67.

[18] S. Aryuyuen, W. Bodhisuwan, T. Ngamkham, Power Garima-generated family of distributions: properties and application, Lobachevskii J. Math., **42,** no. 2, (2021), 287–299.

[19] S. Aryuyuen, W. Bodhisuwan, The type II Topp Leone-power Lomax distribution with analysis in lifetime data, J. Stat. Theory Pract., **14,** no. 2, 1–19.

[20] T. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, D. E. Knuth, On the Lambert W function, Adv. Comput. Math., **5,**, (1996), 329–359.

[21] D. Veberic. Lambert W function for applications in physics, Comput. Phys. Commun., **183,** no. 12, (2012), 2622–2628.

[22] S. Nadarajah, S. Kotz, The beta exponential distribution, Reliab. Eng. Syst. Saf., **91**, no. 6, (2006), 689–697.

[23] R Core Team, R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria, URL https://www.R-project.org/, 2022.

[24] E. T. Lee, Statistical methods for survival data analysis, 2nd Edition, John Wiley and Sons Inc., New York, USA, 1992.

[25] K. Xu, M. Xie, L. C. Tang, S. L. Ho, Application of neural networks in forecasting engine systems reliability, Appl. Soft Comput., **2,** no. 4 (2003), 255–268.