International Journal of Mathematics and
Computer Science, **17**(2022), no. 4, 1483–1489

On the Diophantine equation $7^x + 5 \cdot p^y = z^2$ where $p \equiv 1, 2, 4 \pmod 7$

Wipawee Tangjai[1,2], **Suveera Chaeoueng**[1], **Naruemon Phumchaichot**[1]

[1]Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham, Thailand

[2]Data Innovation and Public Policy Engineering Research Unit
Mahasarakham University
Maha Sarakham, Thailand.

email: wipawee.t@msu.ac.th

(Received April 2, 2022, Accepted May 16, 2022)

Abstract

In this paper, we study the Diophantine equation $7^x + 5p^y = z^2$, where p is a prime number. We show that if $p \equiv 1, 2, 4 \pmod 7$, then the equation has no non-negative integer solution (x, y, z). We also give a necessary condition of the solution, if any, for the case $p \equiv 3, 5, 6 \pmod 7$ when x is even.

1 Introduction

In 1844, Catalan [3] conjectured that $(a, b, x, y) = (3, 2, 2, 3)$ is the unique solution for the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$. The conjecture was proved by Mihăilescu [8] in 2004. Several Diophantine equations in the form $a^x + b^y = z^2$, when a, b are positive integers are x, y and z are non-negative interger have been

---

Key words and phrases: Diophantine equation, congruence, prime number
AMS (MOS) Subject Classifications: 11J99.
The corresponding author is Wipawee Tangjai.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

investigated. In 2011, Suvarnamani, Singta and Chotohaisthit [1] showed that the Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution.

In 2013, Abu Muriefah and AL-Rashed [4] gave the following results.

Theorem 1.1. *[4] If $p \neq 7$, x is an even integer and $(h, p) = 1$, where h is the class number of the field $(\sqrt{-7p})$, then the Diophantine equation*

$$px^2 + 7^{2m+1} = y^p,$$

has no solution in integers x and y.

Theorem 1.2. *[4] Let p be an odd prime such that $p - 7$ has no perfect square.*

- *The Diophantine equation*

  $$x^2 + 7 = py^{p-1},$$

  has no solution in rational x and y such that $y = \frac{Y}{t}$ where Y is an odd integer.

- *The Diophantine equation*

  $$x^2 + 7 = \frac{py^{(p-1)}}{2} \text{ where } p \equiv 1 \pmod 4$$

  has no solution in rational x and y such that $y = \frac{Y}{t}$, where Y is an odd integer.

In 2013, Chotchaisthit [9] showed that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution which is $(x, y, z) = (3, 0, 3)$. In 2014, Sroysang [2] showed that the Diophantine equations $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$ have no non-negative integer solution. In 2019, Burshtein [5] showed that

1. the Diophantine equation $5^x + 103^y = z^2$, has no solution, when x, y and z are positive integers.

2. $(1, 1, 4), (2, 1, 6)$ and $(5, 1, 56)$ are the only three positive integer solutions (x, y, z) for $5^x + 11^y = z^2$, when x, y and z are positive integers.

In 2020, Burshtein [6] showed that

1. $(2, 1, 3)$ is the unique positive integer solution (x, y, z) for $2^x + 5^y = z^2$, where x, y and z are positive integers,

2. the Diophantine equation $7^x + 11^y = z^2$ has no solutions when x, y and z are positive integers.

In 2020, Orosram and Comemuang [11] showed that the Diophantine equation $8^x + n^y = z^2$ has a unique non-negative integer solution which is $(x, y, z) = (1, 0, 3)$. In 2021, N. Viriyapong and C. Viriyapong [7] showed that the Diophantine equation $n^x + 13^y = z^2$ has exactly one solution $(n, x, y, z) = (2, 3, 0, 3)$, where x, y and z are non-negative integers and n is a positive integer with $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square number. In 2022, Orosram and Unchai [10] gave a solution of the Diophantine equation $2^{2nx} - p^y = z^2$, where p is a prime.

In this work, we show that for a prime number $p \neq 7$, the Diophantine equation

$$7^x + 5 \cdot p^y = z^2 \tag{1.1}$$

has no non-negative integer solution (x, y, z) if $p \equiv 1, 2, 4 \pmod 7$. We also give a necessary condition for the solution in the case $p \equiv 3, 5, 6 \pmod 7$ when x is even.

2 Main results

In 2021, Tangjai and Chubthaisong [12] gave the following results.

Theorem 2.1. *[12] Let* p *and* q *be odd prime numbers, where* $p \neq q$*. The solution for* $q^x + p^y = z^2$ *exists only if one of the following is true:*

- $q \equiv 1 \pmod 4, p \equiv 3 \pmod 4$ *and* y *is and odd number,*

- $q \equiv 3 \pmod 4, p \equiv 1 \pmod 4$ *and* x *is an odd number,*

- $q \equiv 3 \pmod 4, p \equiv 3 \pmod 4$ *and the parity of* x, y *is different.*

By considering $7^x + 5 \cdot p^y \equiv z^2 \pmod 4$, the method in Theorem 2.1 allows us to state the following remark.

Remark 2.2. *The Diophantine equation* $7^x + 5 \cdot p^y = z^2$ *has a non-negative integer solution only if one of the following is true*

- x *is an odd number and* $p \equiv 1 \pmod 4$*,*

- $p \equiv 3 \pmod 4$ *and the* $x \not\equiv y \pmod 2$.

Lemma 2.3. *The Diophantine equation* $1 + 5 \cdot p^y = z^2$ *has no non-negative integer solution* (y, z) *where* p *is a prime number.*

Proof. Suppose, to get a contradiction, that there exist non-negative integers y and z, where $1 + 5 \cdot p^y = z^2$. We have $5 \cdot p^y = (z-1)(z+1)$.

There exist non-negative integers α and β such that $\alpha \leq \beta$, $\alpha + \beta = y$ and either

$$p^\alpha = z - 1 \text{ and } 5 \cdot p^\beta = z + 1$$

or

$$5 \cdot p^\beta = z - 1\text{and } p^\alpha = z + 1.$$

We then consider two cases.
Case I : $p^\alpha = z - 1$ and $5 \cdot p^\beta = z + 1$. We have $2 = p^\alpha(5 \cdot p^{\beta-\alpha} - 1)$ which is not possible because $5 \cdot p^{\beta-\alpha} - 1 \geq 4$.

Case II : $5 \cdot p^\alpha = z - 1$ and $p^\beta = z + 1$. We have $2 = p^\alpha(p^{\beta-\alpha} - 5)$. If $p = 2$, then $p^{\beta-\alpha} \geq 2^3$; otherwise, $p^{\beta-\alpha} - 5$ is negative. Hence $p^{\beta-\alpha} - 5 \geq 3$ which is not possible.

Therefore, the Diophantine equation $1 + 5 \cdot p^y = z^2$ has no solution for all prime p. □

Theorem 2.4. *Let* p *be a prime number such that* $p \equiv 1, 2, 4 \pmod 7$. *The Diophantine equation* $7^x + 5 \cdot p^y = z^2$ *has no non-negative integer solution (x,y,z).*

Proof. Suppose, to get a contradiction, that there exist non- negative integers x, y, z such that $7^x + 5 \cdot p^y = z^2$. It follows that z is even and $z^2 \equiv 0, 1, 2, 4 \pmod 7$. Since $p \equiv 1, 2, 4 \pmod 7$, it follows that $p^y \equiv 1, 2, 4 \pmod 7$ for all non-negative integer y. Hence $5 \cdot p^y \equiv 3, 5, 6 \pmod 7$. If $x = 0$, then, $1 + 5 \cdot p^y = z^2$ has no non-negative integer solution by Lemma 2.3. So we consider $x \geq 1$. Since $5 \cdot p^y \equiv 3, 5, 6 \pmod 7$, it follows that $7^x + 5 \cdot p^y \equiv 3, 5, 6 \pmod 7$ which contradicts $z^2 \equiv 0, 1, 2, 4 \pmod 7$. Therefore, the Diophantine equation $7^x + 5 \cdot p^y = z^2$, where $p \equiv 1, 2, 4 \pmod 7$, has no non-negative integer solution (x, y, z). □

Theorem 2.5. *Let* p *be a prime number such that* $p \equiv i \pmod 7$, *for some* $i = 3, 5, 6$. *If* $7^x + 5 \cdot p^y = z^2$ *has a non-negative integer solution, then* y *is an odd number.*

Proof. Since $p \equiv 3, 5, 6 \pmod 7$, it follows that $p^{2k} \equiv 1, 2, 4 \pmod 7$ for all non-negative integer k. Hence $z^2 \equiv 7^x + 5p^y \equiv 3, 4, 6 \pmod 7$. Therefore, there is no solution when y is an even number. □

By Remark 2.2, we have the following result.

Remark 2.6. *The Diophantine equation* $7^x + 5 \cdot p^y = z^2$ *has a non-negative integer solution only if* y *is odd and one of the following is true:*

- x *is an odd number and* $p \equiv 1 \pmod 4$*,*
- x *is an even number and* $p \equiv 3 \pmod 4$*.*

Theorem 2.7. *Let* p *be a prime number such that* $p \equiv 3, 5, 6 \pmod 7$ *and* $p \neq 7$*. For a positive even number* x*, if* $7^x + 5 \cdot p^y = z^2$ *has a non-negative integer solution, then* $p \equiv 3, 5 \pmod 7$ *and* $y \equiv 1, 5 \pmod 6$*.*

Proof. We note that $p^y \equiv 3, 5, 6 \pmod 7$ for all $p \equiv 3, 5, 6 \pmod 7$ and a positive odd number y. Let $x = 2l$ for some positive integer l. We have $5p^y = z^2 - 7^{2l} = (z - 7^l)(z + 7^l)$. Thus, there exist non-negative integer α, β where $\alpha + \beta = y$ such that either one of the following is true:

- $p^\alpha = z - 7^l$ and $5p^\beta = z + 7^l$
- $p^\alpha = z + 7^l$ and $5p^\beta = z - 7^l$.

Case 1: $p^\alpha = z - 7^l$ and $5p^\beta = z + 7^l$. If $p = 3$, then $3^\alpha < 5 \cdot 3^\beta$. Hence $\alpha \leq \beta + 2$; otherwise, $\alpha < \beta + 1$. We first consider $\alpha \leq \beta$. We have $2 \cdot 7^l = p^\alpha(5p^{\beta-\alpha} - 1)$. Since p is an odd prime number that is not 7, it follows that $\alpha = 0$. Thus $2 \cdot 7^l = 5p^\beta - 1 = 5 \cdot p^y - 1$ which is possible only when $p^y \equiv 3 \pmod 7$. Thus, either $p \equiv 3 \pmod 7$ and $y \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 7$ and $y \equiv 5 \pmod 6$. Now, let us consider the case $\alpha = \beta + 1$ or $\alpha = \beta + 2$. In this case, we have $p = 3$. If $\alpha = \beta + 1$, then $2 \cdot 7^l = 5 \cdot 3^\beta - 3^{\beta+1} = 3^\beta(5 - 3) = 2 \cdot 3^\beta$ which is not possible. If $\alpha = \beta + 2$, then $2 \cdot 7^l = 5 \cdot 3^\beta - 3^{\beta+2} = 3^\beta(5 - 9) = -4 \cdot 3^\beta$ which is not possible.

Case 2: $p^\alpha = z + 7^l$ and $5p^\beta = z - 7^l$. We have that $5p^\beta = z - 7^l < z + y^l = p^\alpha$. Hence $\beta < \alpha$ and $2 \cdot 7^l = p^\alpha - 5p^\beta = p^\beta(p^{\alpha-\beta} - 5)$. Since p is an odd prime number that is not 7, it follows that $\beta = 0$ and $2 \cdot 7^l = p^\beta - 5 = p^y - 5$ which is possible only if $p^y \equiv 5 \pmod 7$. Thus, either $p \equiv 3 \pmod 7$ and $y \equiv 5 \pmod 6$ or $p \equiv 5 \pmod 7$ and $p \equiv 1 \pmod 6$. ∎

Theorem 2.8. *Let* p *be a prime number such that* $p \neq 7$*. For an even number* x*, if* $7^x + 5 \cdot p^y = z^2$ *has a non-negative integer solution, then* $p \equiv 3, 19 \pmod{28}$ *and* $y \equiv 1, 5 \pmod 6$*.*

Proof. Suppose that there exists a non-negative solution (x, y, z) of $7^x + 5 \cdot 2^y = z^2$ where x is an even number. By Theorem 2.4, we have $p \not\equiv 1, 2, 4 \pmod 7$. By Lemma 2.3 and Theorem 2.7, we can conclude that the solution for $7^x + 5 \cdot p^y = z^2$ exists only if $p \equiv 3, 5 \pmod 7$. We note that, for $x \geq 1$,

$$7^x \equiv \begin{cases} 7 \pmod{28} & \text{if } x \text{ is odd,} \\ 21 \pmod{28} & \text{if } x \text{ is even,} \end{cases}$$

and $z^2 \equiv 0, 4, 8, 16 \pmod{28}$ when z is even. By the Chinese remainder theorem and Remark 2.6, the solutions exists only if p, x and y satisfy Table 1. □

$p \pmod{28}$	x	y
3	even	$1, 5 \pmod 6$
19	even	$1, 5 \pmod 6$

Table 1: A necessary condition of p, x and y for the existence of the solution of $7^x + 5 \cdot p^y = z^2$.

3 Conclusion

In conclusion, we show that if the solution for the Diophantine equation $7^x + 5 \cdot p^y = z^2$ exists, then $p \equiv 3, 5, 6 \pmod 7$. Moreover, if x is even, then the solution exist only if $p \equiv 3, 19 \pmod{28}$ and $y \equiv 1, 5 \pmod 6$.

Acknowledgment. This research project was financially supported by Mahasarakham University.

References

[1] Alongkot Suvarnamani, Akarate Singta, Somchit Chotchaisthit, On the Diophantine equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, Science and Technology RMUTT Journal, **1,** (2011), 25–28.

[2] Banyat Sroysang, On the Diophantine equation $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$, Int. J. Pure Appl. Math., **92,** (2014), 113–116.

[3] Eugène Charles Catalan, Note extraite dune lettre adressée à léditeur, J. reine angew. Mathematik, **27,** (1844), 192.

[4] Fadwa S. Abu Muriefah, Amal AL-Rashed, On the Diophantine equation $ax^2 + b = cy^n$, J. Anal. Num. Theo., **1,** (2013), 15–21.

[5] Nechemia Burshtein, On solutions to the Diophantine equation $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ with positive integers x, y, z, Annals of Pure and Applied Mathematic, **19,** (2019), 75–77.

[6] Nechemia Burshtein, On the Diophantine equation $2^x + 5^y = z^2$ and $7^x + 11^y = z^2$, Annals of Pure and Applied Mathematics, **21,** (2020), 63–68.

[7] Nongluk Viriyapong, Chokchai Viriyapong, On the Diophantine equation $n^x + 13^y = z^2$, where $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square number, WSEAS Transactions on Mathermatics, **20,** (2021), 442–445.

[8] Preda Mihăilescu, Primary cycolotomic units and a proof of Catalan's conjecture, J. Reine Angew. Math., **572,** (2004), 167–195.

[9] Somchit Chotchaisthit, On the Diophantine equation $2^x + 11^y = z^2$, Maejo Int. J. Sci. Tech., **7,** (2013), 291–293.

[10] Wachirarak Orosram, Ariya Unchai, On the Diophantine equation $2^{2nx} - p^y = z^2$, where p is a prime, International Journal of Mathematics and Computer Science, **17,** no. 1, (2022), 447–451

[11] Wachirarak Orosram, Chalermwut Comemuang, On the Diophantine equation $8^x + n^y = z^2$, WSEAS Transactions on Mathermatics, **19,** (2020), 520–522.

[12] Wipawee Tangjai, Chusak Chubthaisong, On the Diophantine equation $3^x + p^y = z^2$ where $p \equiv 2 \pmod 3$, WSEAS Transactions on Mathematics, **20,** (2021), 283–287.